Final Exam KEY
Mathematical Economics

Instructions: First of all, have fun. Second, as you work through this exam you should be saying to yourself, "oh yeah—COOL—I can see the woman in the red dress!" If not, maybe you will in a couple of days when it's too late. Just kidding! Good luck.

1. If $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ where $\mathbf{y}$ is a vector of n rows and $\mathbf{X}$ is a matrix with k columns and n rows, how many rows and columns must $\boldsymbol{\beta}$ and $\mathbf{e}$ have?

   a. Show that $\mathbf{e}^T\mathbf{e}$ is equal to

$$\mathbf{y}^T\mathbf{y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$$

$$\begin{aligned}\mathbf{e}^T\mathbf{e} &= (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\ &= (\mathbf{y}^T - \boldsymbol{\beta}^T\mathbf{X}^T)(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\ &= \mathbf{y}^T\mathbf{y} - \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} - \mathbf{y}^T\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} \\ &= \mathbf{y}^T\mathbf{y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}\end{aligned}$$

   b. Let $\mathbf{y} = \begin{pmatrix} 6 \\ 5 \\ 8 \\ 3 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} 1 & 10 \\ 1 & 11 \\ 1 & 9 \\ 1 & 12 \end{pmatrix}$, and $\boldsymbol{\beta} = \begin{pmatrix} a \\ b \end{pmatrix}$

   Compute the following: (NOTE: $b_{1,1}$ = a and $b_{2,1} = b$)

   i. $\mathbf{y}^T\mathbf{y}$

$$yty := 134.$$

   ii. $\mathbf{X}^T\mathbf{y}$

$$Xty := \begin{bmatrix} 22. \\ 223. \end{bmatrix}$$

   iii. $\boldsymbol{\beta}^T\mathbf{X}^T$

$$BtXt := [b_{1,1} + 10.b_{2,1} \quad b_{1,1} + 11.b_{2,1} \quad b_{1,1} + 9.b_{2,1} \quad b_{1,1} + 12.b_{2,1}]$$

   iv. $\mathbf{X}\boldsymbol{\beta}$

$$XB := \begin{bmatrix} b_{1,1} + 10.b_{2,1} \\ b_{1,1} + 11.b_{2,1} \\ b_{1,1} + 9.b_{2,1} \\ b_{1,1} + 12.b_{2,1} \end{bmatrix}$$

   v. $\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y}$

$$BtXty := 22.b_{1,1} + 223b_{2,1}$$

vi. $\mathbf{y}^T\mathbf{X}\boldsymbol{\beta}$

Same as part (v)

vii. $\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$

$$BtXtXB = 4.b_{1,1}^{2} + 84.b_{1,1}\,b_{2,1} + 446b_{2,1}^{2}$$

viii. $\mathbf{y}^T\mathbf{y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$

$$ete := 134. - 44.b_{1,1} - 446b_{2,1} + 4.b_{1,1}^{2} + 84.b_{1,1}\,b_{2,1} + 446b_{2,1}^{2}$$

c. Explain why the result in part (b.viii) is the Sum of Squared Residuals (SSR).

$$\mathbf{e}^T\mathbf{e} = \sum_{i=1}^{n} e_i^2$$

d. Compute the 1st order partial derivatives of the SSR (differentiate the SSR with respect to *a* and *b*).

$$\tfrac{\partial e^T e}{\partial a} = -33 + 8a + 84b$$

$$\tfrac{\partial e^T e}{\partial b} = -446 + 84a + 892b$$

e. Compute all four 2nd order partial derivatives of the SSR.

$$SOC1 := 8.$$

$$SOC2 := 84.$$

$$SOC3 := 84.$$

$$SOC4 := 892.$$

f. Solve the 1st order partial derivatives of the SSR for a^* and b^*.

$$b^* = -1.6$$
$$a^* = 22.3$$

g. Is the Hessian in part (g) positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite? **Explain**. So does a^* and b^* minimize or maximize the SSR? **Explain**.

$$H := \begin{bmatrix} 8. & 84. \\ 84. & 892. \end{bmatrix}$$

Since both eigenvalues are both positive, *H* is positive definite and a^* and b^* minimize the SSR

h. If the second column of **X** represents prices of fresh lobster from the ocean for the corresponding quantities given in **y**, provide an economic interpretation of b^*.

A $1 increase in the price of lobster is associated with 1.6 decrease in the consumption of lobster

2. Let L and K denote the quantity of labor and capital, respectively, that is used in the production of Prince Akeem Beef Jerky (what can I say, I like the movie Coming to America) per hour. Let w and r denote the cost of labor and capital, respectively, per hour. Assume that L and K are combined to produce q (units of production are in number of slabs of beef jerky per hour) according to the following technology:

$$q = 100L^{0.5} + 2000K^{0.5}$$

Assume the beef jerky market is perfectly competitive (i.e., Prince Akeem Beef Jerky is a price taker, meaning it cannot restrict output to raise the price in the beef jerky market). Let p denote the price of beef jerky. Prince Akeem Beef Jerky's per hour profit function is

$$\pi = p \cdot \left(100L^{0.5} + 2000K^{0.5}\right) - wL - rK$$

a. Compute the first order partial derivatives of π.

$$\frac{\partial \pi}{\partial L} = \frac{50p}{L^{0.5}} - w$$

$$\frac{\partial \pi}{\partial L} = \frac{1000p}{K^{0.5}} - r$$

b. Compute all 4 second order partial derivatives of π and form the Hessian matrix for this problem. **DO NOT COMPUTE** the eigenvalues.

i. Imagine you are on the surface of π at the optimal solution (K^* and L^*). If L's eigenvalue is negative, what happens to the height π if L is increased or decreased by one holding K constant at K^*?

Stepping in either direction of L means you will be at a lower point on the surface

ii. Again, imagine you are on the surface of π at the optimal solution (K^* and L^*). If L's eigenvalue is positive, what happens to the height π if L is increased or decreased by one holding K constant at K^*?

Stepping in either direction of L means you will be at a higher point on the surface

iii. Should both eigenvalues be negative or positive?

Both eigenvalues should be negative

c. Solve the system of first order partial derivatives for L^* and K^*.

$$L := \frac{2500p^2}{w^2} \qquad K = \frac{1000000\,p^2}{r^2}$$

d. Differentiate K^* with respect to *r*. Differentiate L^* with respect to *w*. Provide an economic interpretation of both of these partial derivatives. Explain why the functions K^* and L^* represent Prince Akeem Beef Jerky's factor (inputs to production) demands.

$$\frac{\partial K}{\partial r} = \frac{-2000000p^2}{r^3} < 0$$

$$\frac{\partial L}{\partial w} = \frac{-5000p^2}{w^3} < 0$$

A negative partial means that if the price of a factor of production increases, the firm will hire less of that factor, holding all else constant; i.e., the law of demand holds for factors of production.

e. Substitute L^* and K^* into *q* to get $q^* = q(L^*, K^*)$. Differentiate this result, with respect to *p*. Provide an economic interpretation of this partial derivative. Explain why $q^* = q(L^*, K^*)$ represent Prince Akeem Beef Jerky's supply curve?

$$q = \frac{5000p}{w} + \frac{2000000p}{r}$$

$$\frac{\partial q}{\partial p} = \frac{5000}{w} + \frac{2000000}{r} > 0$$

An increase in the price of the firm's good means the firm will supply more of that good, holding all else constant ; i.e., the law of supply holds

f. Suppose the labor union representing Prince Akeem Beef Jerky's demands that the firm pay its employees more ($dw > 0$). If the firm agrees to this demand, will it increase or decrease its output of beef jerky slabs? (Hint: differentiate q^* with respect to *w*.)

$$\frac{\partial q}{\partial w} = \frac{-5000p}{w^2} < 0$$

If the wage the firm must pay its employees increases, the firm will scale back production, holding all else constant.

3. Assume inflation (rate of change of prices) is given by

$$\dot{p} = 2(q_d - q_s)$$

where q_s and q_d represent the supply and demand for the economy's good given by

$$q_d = 2000 - 10p$$
$$q_s = 10p$$

where q is economic output and p is price at time t.

a. Compute the equilibrium price and quantity of the economy's good.

$$q_d = q_s$$
$$2000 - 10p = 10p$$
$$2000 = 20p$$
$$p = 100$$
$$q = 1000$$

b. Substitute the right hand sides of q_d and q_s into $\dot{p} = 2(q_d - q_s)$ and then rearrange this result so that it is in the general form of the **linear, autonomous, first-order differential equation** (LAFODE) as given in Definition 21.1 of the text.

$$\dot{p} = 2(2000 - 10p - 10p)$$

$$\dot{p} = 4000 - 40p$$

$$\dot{p} + 40p = 4000$$

c. What is a in Definition 21.1 equal to? What is b equal to? According to Theorem 21.3 of the text, what is the solution to the LAFODE in part (b)?

$$a = 40$$
$$b = 4000$$
$$p(t) = Ce^{-40t} + 100$$

d. If time goes to infinity, what does the solution in part (c) converge to? Using your economic intuition, explain why this is the same as the equilibrium price you computed in part (a).

$$\lim_{t \to \infty} p(t) = 100$$

If $q_d - q_s$ is negative, it represents a shortage but a surplus if positive. Over time either the surplus or shortage will diminish until the market clears

e. Suppose at time $t = 0$ the price of the economy's good is $p = 50$. (This is called the initial condition.) What is the constant C equal to in part (c)?

$$50 = p(0) = Ce^{-40 \cdot 0} + 100$$

$$50 = C + 100$$

$$C = -50$$

$$p(t) = -50e^{-40t} + 100$$

f. How much time elapses before the price of the good is within 50 cents of the equilibrium price computed in part (a)?

$$-50e^{-40t} + 100 = 99.5$$

$$100 - 99.5 = 50e^{-40t}$$

$$0.5 = 50e^{-40t}$$

$$-4.6052 = \ln\left(\tfrac{0.5}{50}\right) = \ln\left(e^{-40t}\right) = -40t$$

$$t = 0.115$$

Extra Credit. Given the **X** and **y** in 1.b, compute the following

1. $\mathbf{X}^T\mathbf{X}$

$$XtX := \begin{bmatrix} 4 & 42. \\ 42. & 446. \end{bmatrix}$$

2. $(\mathbf{X}^T\mathbf{X})^{-1}$

$$invXtX := \begin{bmatrix} 22.30000000 & -2.10000000 \\ -2.100000000 & 0.200000000 \end{bmatrix}$$

3. $\mathbf{X}^T\mathbf{y}$ (Hint: You already computed this!)
4. $(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$

$$B := \begin{bmatrix} 22.3000000 \\ -1.6000000 \end{bmatrix}$$

5. Compare the computation in (4) to those made in question 1 part (f).

They are the same.

6. Are you seeing the woman in the red dress yet? Expound.