

COMP 785 Advanced Algorithms Spring 2007 Exam 2

100 points total

- 1 (16 points). Three people independently choose (without bias) a number between 1 and m . What is the probability that they all choose the same number? Justify your answer.

Answer

We have a uniform probability distribution here, with the probability of any person choosing a given number between 1 and m being $1/m$.

Arbitrarily label the people a , b , and c .

Suppose that a chooses x , $1 \leq x \leq m$. This is the event of choosing some number between 1 and m from the numbers between 1 and m and is the certain event, with probability 1. The events of b choosing x and c choosing x are independent, and each has probability $1/m$. So the probability of all three choosing x is $1/m \cdot 1/m = 1/m^2$. This is true for any number x , $1 \leq x \leq m$, so the answer is $1/m^2$.

Alternatively, the probability of all three choosing 1 is $1/m \cdot 1/m \cdot 1/m = 1/m^3$, of all three choosing 2 is $1/m^3$, ..., of all three choosing m is $1/m^3$. Since these composite events are all independent, the probability of any one of these happening is the sum of the individual probabilities, viz., $m(1/m^3) = 1/m^2$.

- 2 (22 points). Suppose we have n servers and 3 clients. In a given hour, each client sends at most one request to a given server (but it may send requests to any number of the n servers during the hour). The probability that a client sends a request to a given server is independent of the server (i.e., it is just as likely to send a request to one server as to another). Calling the clients α , β , and γ , the probability of sending a request during the hour to a given server for each of the clients is 0.5 for α , 0.6 for β , and 0.4 for γ . Define the indicator random variable $X_{i,j}$ as:

$$X_{i,j} = I\{\text{client } i \text{ sends a request during the hour to server } j\}$$

Let X be the random variable denoting the number of requests sent (from any client to any server) during a given hour. Using the indicator random variables $X_{i,j}$, find the expected value of X . You might find it helpful to note that, where ϕ is any expression,

$$\sum_{i=1}^m \sum_{j=1}^n \phi = \sum_{j=1}^n \sum_{i=1}^m \phi$$

Answer

To facilitate indexing, refer to client α as "1", client β as "2", and client γ as "3."
Then

$$X = \sum_{j=1}^n \sum_{i=1}^3 X_{i,j}$$

So

$$E[X] = E\left[\sum_{j=1}^n \sum_{i=1}^3 X_{i,j}\right] = \sum_{j=1}^n \sum_{i=1}^3 E[X_{i,j}]$$

The last step is justified by the linearity of expectation. Now, by Lemma 5.1 in the text,

$$E[X_{1,j}] = \Pr\{\text{client 1 sends a request during the hour to server } j\} = 0.5$$

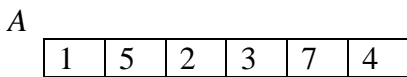
$$E[X_{2,j}] = \Pr\{\text{client 2 sends a request during the hour to server } j\} = 0.6$$

$$E[X_{3,j}] = \Pr\{\text{client 1 sends a request during the hour to server } j\} = 0.4$$

So we have, further,

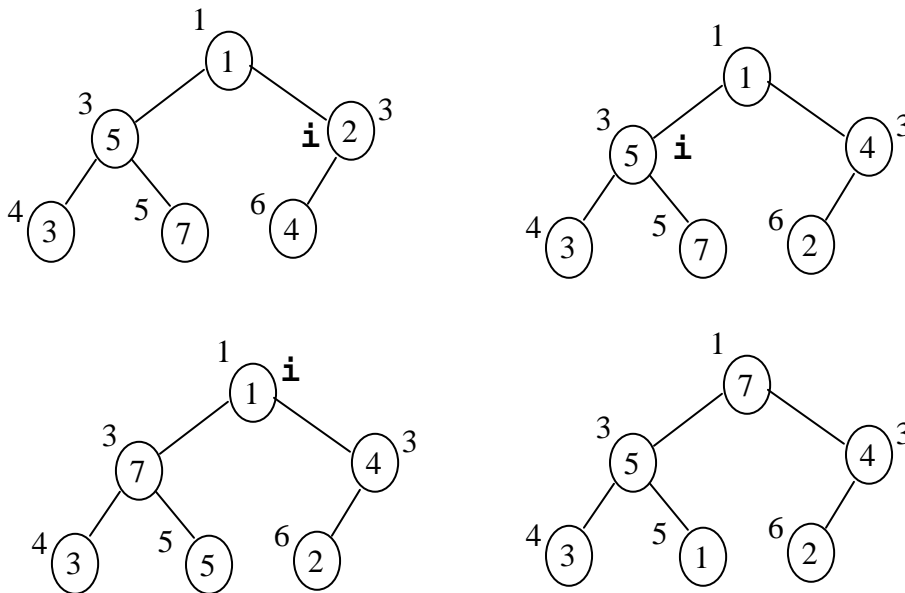
$$E[X] = \sum_{j=1}^n (0.5 + 0.6 + 0.4) = \sum_{j=1}^n 1.5 = 1.5n$$

3 (22 points). Using Figure 6.3 on p. 134 of your text as a model, show the operation of BUILD-MAX-HEAP on the following array.



Answer

The order here is top-down, left-to-right.



4 (18 points). Using Figure 8.3 (p. 171 of your text), show the operation of radix sort on the following list of six 3-digit numbers.

- 555
- 247
- 841
- 291
- 818
- 558

Answer

555	841	818	247
247	291	841	291
841	555	247	555
291	⇒ 247	⇒ 555	⇒ 558
818	818	558	818
588	558	291	841

5 (22 points). Suppose we use RANDOMIZED-SELECT to select the third smallest element of the array $A = \langle 5, 2, 9, 0, 7 \rangle$. Describe a sequence of partitions that results in a best-case performance of RANDOMIZED-SELECT. (In RANDOMIZED-PARTITION, you choose the i selected by the call to RANDOM. Note that $A[p]$ and $A[i]$ are swapped before PARTITION(A, p, r) is called.)

Answer

- This assumes the version of PARTITION and RANDOMIZED-SELECT given in the class notes. If you used the versions from the text, I'll be able to tell and will grade accordingly.

In RANDOMIZED-SELECT, choose $i = 1$ so that A remains $\langle 5, 2, 9, 0, 7 \rangle$ ($p = 1, r = 5$), and the pivot is 5. Having the third smallest element as the pivot lets us isolate the third smallest element as quickly as possible. PARTITION swaps 5 and 0, giving

$$\langle 0, 2, 9, 5, 7 \rangle.$$

Next, $j = 2$ and $i = 3$, so $i < j$ is false, and we exit RANDOMIZED-PARTITION, returning $q = 2$. Since we seek the third smallest element, we retain only the right side of the partition. Since we have eliminated the smallest two elements, we seek the $(3-2)$ st = first smallest element in this subarray.

The subarray we're now considering is

$$\langle 9, 5, 7 \rangle \quad p = 3, r = 5$$

To isolate the smallest element in this subarray, we choose $i = 4$ to make 5, the smallest element, the pivot. So we swap the elements at indices 3 ($= p$) and 4 ($= i$), giving the subarray

$$\langle 5, 9, 7 \rangle.$$

PARTITION does no swaps, and ends with $i = j = 3$. So RANDOMIZED-PARTITION returns $q = 3$. Since we seek the smallest element in this subarray, we retain only the left side of the partition, viz.,

$$\langle 5 \rangle \quad p = 3, r = 5$$

Since $p = r$, the algorithm terminates, returning $A[3] = 5$.