COMP 681 Formal Methods Spring 2008 Recitation 4—Solutions

1. Consider the following conditional. Note that the conditions of the conditionals within the {…} are mutually exclusive.

```
if P and Q then  {
  if R the S1
  else S2
}
else if T then  {
   if Q then S1
   else S2
}
else S2
```

a. In the language of propositional logic, write the conditions under which each of the statements S_1 and S_3 is executed. Simplify, at least move all ¬'s inward so they apply only to primes and factor out common terms.

Answer

$\underline{S_1}$

$P \wedge Q \wedge R \vee \neg (P \wedge Q) \wedge T \wedge Q$	
$\langle\equiv\rangle\ P \wedge Q \wedge R \vee (\neg P \vee \neg Q) \wedge T \wedge Q$	De Morgan's 1st Law
$\langle\equiv\rangle\ P \wedge Q \wedge R \vee \neg P \wedge T \wedge Q \vee \neg Q \wedge T \wedge Q$	Distributive Law ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ P \wedge Q \wedge R \vee \neg P \wedge T \wedge Q \vee T \wedge Q \wedge \neg Q$	Commutativity of $\wedge$
$\langle\equiv\rangle\ P \wedge Q \wedge R \vee \neg P \wedge T \wedge Q \vee T \wedge \mathit{false}$	Law of Contradiction
$\langle\equiv\rangle\ P \wedge Q \wedge R \vee \neg P \wedge T \wedge Q \vee \mathit{false}$	Simplification (3.20)
$\langle\equiv\rangle\ P \wedge Q \wedge R \vee \neg P \wedge T \wedge Q$	Simplification (3.21)
$\langle\equiv\rangle\ Q \wedge P \wedge R \vee Q \wedge \neg P \wedge T$	Commutativity of $\wedge$ (2×)
$\langle\equiv\rangle\ Q \wedge (P \wedge R \vee \neg P \wedge T)$	Distributive Law ($\wedge$ over $\vee$)

$\underline{S_2}$

$P \wedge Q \wedge \neg R \vee \neg (P \wedge Q) \wedge T \wedge \neg Q \vee \neg (P \wedge Q) \wedge \neg T$	
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee (\neg P \vee \neg Q) \wedge T \wedge \neg Q \vee \neg (P \wedge Q) \wedge \neg T$	De Morgan's 1st Law
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge T \wedge \neg Q \vee \neg Q \wedge T \wedge \neg Q \vee \neg (P \wedge Q) \wedge \neg T$	Distr. ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge T \wedge \neg Q \vee T \wedge \neg Q \wedge \neg Q \vee \neg (P \wedge Q) \wedge \neg T$	Comm. of $\wedge$
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge T \wedge \neg Q \vee T \wedge \neg Q \vee \neg (P \wedge Q) \wedge \neg T$	Idempotence of $\wedge$
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee T \wedge \neg Q \vee \neg (P \wedge Q) \wedge \neg T$	Absorption (3.22)
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee T \wedge \neg Q \vee (\neg P \vee \neg Q) \wedge \neg T$	De Morgan's 1st Law
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee T \wedge \neg Q \vee \neg P \wedge \neg T \vee \neg Q \wedge \neg T$	Distr. ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge \neg T \vee T \wedge \neg Q \vee \neg Q \wedge \neg T$	Commutativity of $\vee$
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge \neg T \vee \neg Q \wedge T \vee \neg Q \wedge \neg T$	Commutativity of $\wedge$
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge \neg T \vee \neg Q \wedge (T \vee \neg T)$	Distr. ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge \neg T \vee \neg Q \wedge \mathit{true}$	Excluded Middle
$\langle\equiv\rangle\ P \wedge Q \wedge \neg R \vee \neg P \wedge \neg T \vee \neg Q$	Simplification (3.18)

b. Rewrite the conditional in the form

```
if _ then  {
  if _ then  {
    if _ then __
    else __
  }
  else  {
    if _ then __
    else __
}
else __
```

Answer

First modify the expression for S_2 as follows:

$P \wedge Q \wedge \neg R \;\vee\; \neg P \wedge \neg T \;\vee\; \neg Q$	The result above
$\langle\equiv\rangle\; P \wedge Q \wedge \neg R \;\vee\; \neg P \wedge \neg T \wedge true \;\vee\; \neg Q$	Simplification (3.18)
$\langle\equiv\rangle\; P \wedge Q \wedge \neg R \;\vee\; \neg P \wedge \neg T \wedge (Q \vee \neg Q) \vee \neg Q$	Law of Excluded Middle
$\langle\equiv\rangle\; P \wedge Q \wedge \neg R \;\vee\; \neg P \wedge \neg T \wedge Q \;\vee\; \neg P \wedge \neg T \wedge \neg Q \;\vee\; \neg Q$	Distr. ($\wedge$ over $\vee$)
$\langle\equiv\rangle\; P \wedge Q \wedge \neg R \;\vee\; \neg P \wedge \neg T \wedge Q \;\vee\; \neg Q$	Idempotence (3.22)

Using the final expression above for S_1 and the expression for S_2 derived here, we can read off the conditions.

```
if Q then  {
  if P then  {
    if R then S1
    else S2
  }
  else  {
    if T then S1
    else S2
}
else S2
```

2. A group of experts from different departments decides that procedure S_1 is done when P is true, that S_2 is done if R is false, and that S_3 is done if both Q and R hold.

a. Write a GCL alternation statement that captures these requirements. (The conditions won't be mutually exclusive at this point.)

Answer

```
if     P   →    S1
□    ¬ R   →    S2
□  Q ∧ R   →    S3
fi
```

b. You point out to the experts that their conditions are not mutually exclusive. They decide among themselves that S_1 should take priority over the other two statements and that S_3 should take priority over S_2. Rewrite the conditions so that they are mutually exclusive and respect these priorities.

Answer

Before simplification, we have

if $P \rightarrow S_1$

□ $\neg P \wedge \neg (Q \wedge R) \wedge \neg R \rightarrow S_2$

□ $\neg P \wedge Q \wedge R \rightarrow S_3$

fi

We simplify the guard for S_2 as follows.

$\neg P \wedge \neg (Q \wedge R) \wedge \neg R$

$\langle \equiv \rangle$ $\neg P \wedge (\neg Q \vee \neg R) \wedge \neg R$ De Morgan's 1st Law

$\langle \equiv \rangle$ $\neg P \wedge \neg R$ Absorption (3.23)

So the above simplifies to

if $P \rightarrow S_1$

□ $\neg P \wedge \neg R \rightarrow S_2$

□ $\neg P \wedge Q \wedge R \rightarrow S_3$

fi

c. For the cases not covered in **b**, add another alternative whose guard covers the remaining cases yet is disjoint from each of the other three guards. The command associated with this guard should be `skip`.

Answer

We form the conjunction of the negations of the guards and simplify:

$\neg P \wedge \neg(\neg P \wedge \neg R) \wedge \neg(\neg P \wedge Q \wedge R)$	
$\langle\equiv\rangle\ \neg P \wedge (\neg\neg P \vee \neg\neg R) \wedge (\neg\neg P \vee \neg Q \vee \neg R)$	De Morgan's 1st Law (2×)
$\langle\equiv\rangle\ \neg P \wedge (P \vee R) \wedge (P \vee \neg Q \vee \neg R)$	Law of Negation (3×)
$\langle\equiv\rangle\ (\neg P \wedge P \vee \neg P \wedge R) \wedge (P \vee \neg Q \vee \neg R)$	Distributive Law ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ (\mathit{false} \vee \neg P \wedge R) \wedge (P \vee \neg Q \vee \neg R)$	Law of Contradiction
$\langle\equiv\rangle\ \neg P \wedge R \wedge (P \vee \neg Q \vee \neg R)$	Simplification (3.21)
$\langle\equiv\rangle\ \neg P \wedge R \wedge P \vee \neg P \wedge R \wedge \neg Q \vee \neg P \wedge R \wedge \neg R$	Distributive Law ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ \neg P \wedge R \wedge P \vee \neg P \wedge R \wedge \neg Q \vee \neg P \wedge \mathit{false}$	Law of Contradiction
$\langle\equiv\rangle\ \neg P \wedge R \wedge P \vee \neg P \wedge R \wedge \neg Q \vee \mathit{false}$	Simplification (3.20)
$\langle\equiv\rangle\ \neg P \wedge R \wedge P \vee \neg P \wedge R \wedge \neg Q$	Simplification (3.21)
$\langle\equiv\rangle\ R \wedge P \wedge \neg P \vee \neg P \wedge R \wedge \neg Q$	Commutativity of $\wedge$
$\langle\equiv\rangle\ R \wedge \mathit{false} \vee \neg P \wedge R \wedge \neg Q$	Law of Contradiction
$\langle\equiv\rangle\ \mathit{false} \vee \neg P \wedge R \wedge \neg Q$	Simplification (3.20)
$\langle\equiv\rangle\ \neg P \wedge R \wedge \neg Q$	Simplification (3.21)
$\langle\equiv\rangle\ \neg P \wedge \neg Q \wedge R$	Law of Contradiction

So the final alternation statement is

```
if     P              →    S1
□   ¬ P ∧ ¬ R         →    S2
□ ¬ P ∧ Q ∧ R         →   S3
□ ¬ P ∧ ¬ Q ∧ R →  skip
fi
```

d. Translate the alternative statement in **c** into the language used in section 3.3 Minimize the number of primes that must be evaluated.

Answer

```
if P then S1
else  {
  if R then  {
    if Q the S3
    }
    else S2
}
```