COMP 681 Formal Methods Spring 2008 Assignment 6—Solutions

Due Monday, April 28

1. Translate the following into the language of predicate logic using the indicated function symbols.

a. *People who work in the same office have the same boss.*

Use the function symbols *office*(_) and *boss*(_), where

$office(x) = y$ means y is the office in which x works, and

$boss(x) = y$ means y is x's boss.

Answer

$\forall x, y \bullet office(x) = office(y) \Rightarrow boss(x) = boss(y)$

c. *No student at A&T has a higher GPA than John's roommate.*

Use the constant symbols *A&T* and *John* and the function symbols *college_of*(_), *gpa*(_), and *roommate*(_), where

$college\_of(x) = y$ means y is the college x attends,

$gpa(x) = y$ means y is x's GPA, and

$roommate(x) = y$ means y is x's roommate.

Answer

$\forall x \bullet college\_of(x) = A\&T \Rightarrow \neg\ (gpa(x) > gpa(roommate(John)))$

2. Let

$\varphi = \forall x \bullet \exists z \bullet p(z, x, h(y)) \Rightarrow q(x, y)$

$t_1 = f(g(u), v)$

$t_2 = f(y, g(z))$

a. Is t_1 free for y in φ? **Answer**: Yes. (Note that there's only 1 free occurrence of x in φ.)

b. Is t_2 free for y in φ? **Answer**: No. The y would be captured by the $\exists y$ if t2 were substituted for the free occurrence of x in φ.

3. Show that the following arguments are valid.

$\forall x \bullet (\exists y \bullet p(x, y) \wedge q(y)) \Rightarrow f(x) = g(x)$
$p(A, B)$
$q(B)$
$\forall x \bullet p(x, B) \Rightarrow h(f(x), g(A)) = D$

$h(g(A), g(A)) = D$

Answer

1.	$\forall x \bullet (\exists y \bullet p(x, y) \wedge q(y)) \Rightarrow f(x) = g(x)$	premise
2.	$p(A, B)$	premise
3.	$q(B)$	premise
4.	$\forall x \bullet p(x, B) \Rightarrow h(f(x), g(A)) = D$	premise
5.	$(\exists y \bullet p(A, y) \wedge q(y)) \Rightarrow f(A) = g(A)$	from 1, ∀_E
6.	$p(A, B) \wedge q(B)$	from 2 & 3, ∧_I
7.	$\exists y \bullet p(A, y) \wedge q(y)$	from 6, ∃_I
8.	$f(A) = g(A)$	from 5 & 7, ⇒_E (Modus ponens)
9.	$p(A, B) \Rightarrow h(f(A), g(A)) = D$	from 4, ∀_E
10.	$h(f(A), g(A)) = D$	from 9 & 2, ⇒_E (Modus ponens)
11.	$h(g(A), g(A)) = D$	from 8 & 10, =_E

4. Prove the following by simple (weak) induction.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Answer

Basis: $n = 1$

$$LHS = \sum_{i=1}^{1} i^2 = 1^2 = 1$$

$$RHS = \frac{1(1+1)(2 \cdot 1+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 6$$

So $LHS = RHS$

Induction step

Assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

(and show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$)

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by the induction hypothesis}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6}$$

$$= \frac{(k(2k+1) + 6(k+1))(k+1)}{6} = \frac{(2k^2 + 7k + 6)(k+1)}{6} = \frac{(k+2)(2k+3)(k+1)}{6}$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6} \qquad \text{Q.E.D.}$$

5. Prove the following by strong induction.

The largest element in a subtree of a heap is at the root of the subtree. (You are given only the heap condition. “Subtree” here is taken in the sense of a tree that is a [not necessarily proper] part of a tree. [Thus, for example, a heap is a subtree of itself.])

Answer

Assume that elements are distinct and that he heaps are non-empty.
We prove this for heaps. It applies to subtrees of heaps, which themselves are also heaps.

<u>Basis</u>: Consider a singleton heap.

In a singleton heap, the only node is the root.
So largest (and only) element must be at the root.

<u>Induction Step</u>

Assume that, in a heap of size i, $1 \le i \le k$, the largest element is at the root (and show that, for a heap of size k+1, the largest element is at the root).

Let H be a heap of size k +1 with left subtree L and right subtree R. Both L and R are heaps. Since $k \ge 1$, so $k+1 > 1$, L can’t be empty

Since the root is a node, L and R (if it exists) have at most k nodes each, so, by the induction hypothesis, their largest elements are at their roots. By the heap property, the element at the root of H is greater than the element at its left child, the root of L, hence it is greater than any element in L. Again by the heap property, the element at the root of H is greater than its right child (if it exists), the root of R, hence it is greater than any element of R (if there are any). So the root of H is greater than any elements in either subtree, that is, it is the largest element in the heap.

6. You are given the following valid predicate-logic formula:

$\forall x \bullet \varphi(x) \wedge \psi(x)\ \langle\equiv\rangle\ (\forall x \bullet \varphi(x)) \wedge (\forall x \bullet \psi(x))$

By induction, prove that the generalization of this is valid, that is, prove

$\forall x \bullet \varphi_0(x) \wedge \varphi_1(x) \wedge \ldots \wedge \varphi_n(x)\ \langle\equiv\rangle\ (\forall x \bullet \varphi_0(x)) \wedge (\forall x \bullet \varphi_1(x)) \wedge .. \wedge (\forall x \bullet \varphi_n(x))$

Answer

We use weak induction on the number *n* of $\wedge$'s.

Basis: n = 1

This is given by valid formula stated above.

Induction step

Assume the result holds for k ≥ 1 $\wedge$'s:

$\forall x \bullet \varphi_0(x) \wedge \varphi_1(x) \wedge \ldots \wedge \varphi_k(x)\ \langle\equiv\rangle\ (\forall x \bullet \varphi_0(x)) \wedge (\forall x \bullet \varphi_1(x)) \wedge .. \wedge (\forall x \bullet \varphi_k(x))$

(and show that it holds for k+1 $\wedge$'s:

$\forall x \bullet \varphi_0(x) \wedge \varphi_1(x) \wedge \ldots \wedge \varphi_n(x) \wedge \varphi_{k+1}(x)$

$\langle\equiv\rangle\ (\forall x \bullet \varphi_0(x)) \wedge (\forall x \bullet \varphi_1(x)) \wedge .. \wedge (\forall x \bullet \varphi_n(x)) \wedge (\forall x \bullet \varphi_{k+1}(x))$)

$\forall x \bullet \varphi_0(x) \wedge \varphi_1(x) \wedge \ldots \wedge \varphi_n(x) \wedge \varphi_{k+1}(x)$

$\langle\equiv\rangle\ \forall x \bullet (\varphi_0(x) \wedge \varphi_1(x) \wedge \ldots \wedge \varphi_n(x)) \wedge \varphi_{k+1}(x)$ by Assoc. of $\wedge$

$\langle\equiv\rangle\ (\forall x \bullet \varphi_0(x) \wedge \varphi_1(x) \wedge \ldots \wedge \varphi_n(x)) \wedge (\forall x \bullet \varphi_{k+1}(x))$ by the given formula

$\langle\equiv\rangle\ ((\forall x \bullet \varphi_0(x)) \wedge (\forall x \bullet \varphi_1(x)) \wedge .. \wedge (\forall x \bullet \varphi_k(x))) \wedge (\forall x \bullet \varphi_{k+1}(x))$ by the ind. hyp.

$\langle\equiv\rangle\ (\forall x \bullet \varphi_0(x)) \wedge (\forall x \bullet \varphi_1(x)) \wedge .. \wedge (\forall x \bullet \varphi_k(x)) \wedge (\forall x \bullet \varphi_{k+1}(x))$ by Assoc. of $\wedge$

Q.E.D.

7. You are given the following version of the rule Constructive Dilemma:

$p \Rightarrow q$
$r \Rightarrow s$
$p \vee r$

$q \vee s$

By induction, prove the generalization of this:

$p_1 \Rightarrow q_1$
$p_2 \Rightarrow q_2$
…
$p_n \Rightarrow q_n$
$p_1 \vee p_2 \vee \ldots \vee p_n$

$q_1 \vee q_2 \vee \ldots \vee q_n$

Answer

We use weak induction on the number n of premises of the form pi ⇒ qi.

Basis: n = 2

This is a case of the given rule.

Induction step

Assume the rule is valid for k ≥ 2 implicational premises:

$p_1 \Rightarrow q_1$
$p_2 \Rightarrow q_2$
…
$p_k \Rightarrow q_k$
$p_1 \vee p_2 \vee \ldots \vee p_k$

$q_1 \vee q_2 \vee \ldots \vee q_k$

(and show that it holds for k+1 implicational premises:

$p_1 \Rightarrow q_1$
$p_2 \Rightarrow q_2$
…
$p_k \Rightarrow q_k$
$p_{k+1} \Rightarrow q_{k+1}$
$p_1 \vee p_2 \vee \ldots \vee p_k \vee p_{k+1}$

$q_1 \vee q_2 \vee \ldots \vee q_k \vee q_{k+1}$)

We first prove a lemma:

$p_k \Rightarrow q_k, p_{k+1} \Rightarrow q_{k+1} \;|\!- \; p_k \vee p_{k+1} \Rightarrow q_k \vee q_{k+1}$

Proof of the lemma:

1. $p_k \Rightarrow q_k$ Premise
2. $p_{k+1} \Rightarrow q_{k+1}$ Premise
3. $p_k \vee p_{k+1}$ Assumption
4. $q_k \vee q_{k+1}$ 1, 2, & 3 by Constructive Dilemma
5. $p_k \vee p_{k+1} \Rightarrow q_k \vee q_{k+1}$ 3-4, ⇒_I (Conditional Proof)

Now for the main proof:

1. $p_1 \Rightarrow q_1$ Premise

2. $p_2 \Rightarrow q_2$ Premise

…

k. $P_k \Rightarrow q_k$ Premise

k+1. $P_{k+1} \Rightarrow q_{k+1}$ Premise

k+2. $p_1 \vee p_2 \vee \ldots \vee p_k \vee p_{k+1}$ Premise

k+3. $p_1 \vee p_2 \vee \ldots \vee (p_k \vee p_{k+1})$ k+2, Associativity of ∨

k+4. $p_k \vee p_{k+1} \Rightarrow q_k \vee q_{k+1}$ k and k+1, by the above lemma

k+5. $q_1 \vee q_2 \vee \ldots \vee (q_k \vee q_{k+1})$ 1 to k-1, k+4, and k+3, by the ind.hypothesis

k+6. $q_1 \vee q_2 \vee \ldots \vee q_k \vee q_{k+1}$ k+5, Associativity of ∨ Q.E.D