COMP 681 Formal Methods Spring 2008 Assignment 4—<u>Solutions</u>

1. Prove the following using deductive proofs. Justify each step. Show all substitutions used to derive appropriate instances of the rules.

a. $p \wedge q \Rightarrow r\ ,\ p \vee \neg s\ ,\ s \Rightarrow q\ ,\ t \wedge s \quad |\!-\ \ r$

Answer

1. $p \wedge q \Rightarrow r$	premise
2. $p \vee \neg s$	premise
3. $s \Rightarrow q$	premise
4. $t \wedge s$	premise
5. $s \wedge t$	from 4, Commutativity of $\wedge$ $\{ p / t\ ,\ q / s \}$
6. s	from 5, Simplification ($\wedge$_E) $\{ p / s\ ,\ q / t \}$
7. q	from 3 & 6, Modus ponens ($\Rightarrow$_E) $\{ p / s \}$
8. $\neg \neg s$	from 6, Law of negation $\{ p / s \}$
9. $\neg s \vee p$	from 2, Commutativity of $\vee$ $\{ q / \neg s \}$
10. p	from 9 & 8, Disjunctive syllogism ($\vee$_E) $\{ p / \neg s\ ,\ q / p \}$
11. $p \wedge q$	from 10 & 7, Conjunction ($\wedge$_I)
12. r	from 1 & 11, Modus ponens ($\Rightarrow$_E) $\{ p / p \wedge q\ ,\ q / r \}$

b. $p \Rightarrow t \wedge q\ ,\ r \vee p \vee \neg s\ ,\ s \wedge \neg r \quad |\!-\ \ s \wedge t$

Answer

1. $p \Rightarrow t \wedge q$	premise
2. $r \vee p \vee \neg s$	premise
3. $s \wedge \neg r$	premise
4. s	from 3, Simplification ($\wedge$_E) $\{ p / s\ ,\ q / \neg r \}$
5. $\neg r \wedge s$	from 3, Commutativity of $\wedge$ $\{ p / s\ ,\ q / \neg r \}$
6. $\neg r$	from 5, Simplification ($\wedge$_E) $\{ p / \neg r\ ,\ q / s \}$
7. $p \vee \neg s$	from 2 & 6, Disjunctive syllogism ($\vee$_E) $\{ p / r\ ,\ q / p \vee \neg s \}$
8. $\neg s \vee p$	from 7, Commutativity of $\vee$ $\{ q / \neg s \}$
9. $\neg \neg s$	from 4, Law of negation $\{ p / s \}$
10. p	from 8 & 9, Disjunctive syllogism ($\vee$_E) $\{ p / \neg s\ ,\ q / p \}$
11. $t \wedge q$	from 1 & 10, Modus ponens ($\Rightarrow$_E) $\{ q / t \wedge q \}$
12. t	from 11, Simplification ($\wedge$_E) $\{ p / t \}$

2. Prove the following using deductive proofs. Justify each step, but you need not show the substitution used to derive the appropriate instance of the laws. You may use commuted and generalized versions of the rules that involve $\wedge$ and $\vee$.

a. $p \wedge q \vee r \wedge s\ ,\ r \Rightarrow w \vee t\ ,\ t \Rightarrow u\ ,\ \neg (u \vee w)\ ,\ p \Rightarrow (q \Rightarrow v)\ \ \vdash\ \ v \wedge \neg (u \vee r)$

Answer

1. $p \wedge q \vee r \wedge s$	premise
2. $r \Rightarrow w \vee t$	premise
3. $t \Rightarrow u$	premise
4. $\neg (u \vee w)$	premise
5. $p \Rightarrow (q \Rightarrow v)$	premise
6. $\neg u \wedge \neg w$	from 4, De Morgan's 2nd Law
7. $\neg u$	from 6, Simplification ($\wedge$_E)
8. $\neg t$	from 3 & 7, Modus tollens ($\Rightarrow$_E)
9. $\neg w$	from 6, Simplification ($\wedge$_E)
10. $\neg w \wedge \neg t$	from 9 & 8, Conjunction ($\wedge$_I)
11. $\neg (w \vee t)$	from 10, De Morgan's 2nd Law
12. $\neg r$	from 2 & 11, Modus tollens ($\Rightarrow$_E)
13. $\neg r \vee \neg s$	from 12, Addition ($\vee$_I)
14. $\neg (r \wedge s)$	from 13, De Morgan's 1st Law
15. $p \wedge q$	from 1 & 14, Disjunctive syllogism ($\vee$_E)
16. p	from 15, Simplification ($\wedge$_E)
17. $q \Rightarrow v$	from 5 & 16, Modus ponens ($\Rightarrow$_E)
18. q	from 15, Simplification ($\wedge$_E)
19. v	from 17 & 18, Modus ponens ($\Rightarrow$_E)
20. $\neg u \wedge \neg r$	from 7 & 12, Conjunction ($\wedge$_I)
21. $\neg (u \vee \neg r)$	from 20, De Morgan's 2nd Law
22. $v \wedge \neg (u \vee \neg r)$	from 19 & 21, Conjunction ($\wedge$_I)

b. $p \wedge q \Rightarrow r$, $s \wedge t \Rightarrow u$, $p \vee \neg v$, $w \Rightarrow q \wedge s$, $w \vee y \Rightarrow t$, $v \wedge w$ |- $r \wedge u$

Answer

1. $p \wedge q \Rightarrow r$	premise
2. $s \wedge t \Rightarrow u$	premise
3. $p \vee \neg v$	premise
4. $w \Rightarrow q \wedge s$	premise
5. $w \vee y \Rightarrow t$	premise
6. $v \wedge w$	premise
7. v	from 6, Simplification ($\wedge$_I)
8. $\neg\neg v$	from 7, Law of Negation
9. p	from 3 & 8, Disjunctive syllogism ($\vee$_E)
10. w	from 6, Simplification ($\wedge$_E)
11. $q \wedge s$	from 4 & 10, Modus ponens ($\Rightarrow$_E)
12. $w \vee y$	from 10, Addition ($\vee$_I)
13. t	from 5 & 12, Modus ponens ($\Rightarrow$_E)
14. q	from 11, Simplification ($\wedge$_E)
15. s	from 11, Simplification ($\wedge$_E)
16. $p \wedge q$	from 9 & 14, Conjunction ($\wedge$_I)
17. $s \wedge t$	from 15 & 13, Conjunction ($\wedge$_I)
18. r	from 1 & 16, Modus ponens ($\Rightarrow$_E)
19. u	from 2 & 12, Modus ponens ($\Rightarrow$_E)
20. $r \wedge u$	from 18 & 19, Conjunction ($\wedge$_I)

3. Use conditional proof to prove the following.

a. $\neg p \vee (q \Rightarrow t \wedge u) ,\ p \wedge q \Rightarrow v \vee w ,\ \neg v ,\ w \Rightarrow s \ \mid\!-\ p \Rightarrow (q \wedge r \Rightarrow s \wedge t)$

Answer

1. $\neg p \vee (q \Rightarrow t \wedge u)$	premise
2. $p \wedge q \Rightarrow v \vee w$	premise
3. $\neg v$	premise
4. $w \Rightarrow s$	premise
5. p	assumption
6. $q \Rightarrow t \wedge u$	from 1 & 5, Disjunctive syllogism ($\vee$_E)
7. $q \wedge r$	assumption
8. q	from 7, Simplification ($\wedge$_E)
9. $t \wedge u$	from 6 & 8, Modus ponens ($\Rightarrow$_E)
10. t	from 9, Simplification ($\wedge$_E)
11. $p \wedge q$	from 5 & 8, Conjunction ($\wedge$_I)
12. $v \vee w$	from 2 & 11, Modus ponens ($\Rightarrow$_E)
13. w	from 12 & 3, Disjunctive syllogism ($\vee$_E)
14. s	from 4 & 13, Modus ponens ($\Rightarrow$_E)
15. $s \wedge t$	from 14 & 10, Conjunction ($\wedge$_I)
16. $q \wedge r \Rightarrow s \wedge t$	from 7-15, Conditional proof ($\Rightarrow$_I)
17. $p \Rightarrow (q \wedge r \Rightarrow s \wedge t)$	from 5-16, Conditional proof ($\Rightarrow$_I)

b. $p \Rightarrow u ,\ u \Rightarrow q \wedge w ,\ \neg r \vee \neg w \vee s ,\ q \wedge r \Rightarrow t \ \mid\!-\ p \Rightarrow q \wedge (r \Rightarrow s \wedge t)$

Answer

1. $p \Rightarrow u$	premise
2. $u \Rightarrow q \wedge w$	premise
3. $\neg r \vee \neg w \vee s$	premise
4. $q \wedge r \Rightarrow t$	premise
5. p	assumption
6. u	from 1 & 5, Modus ponens ($\Rightarrow$_E)
7. $q \wedge w$	from 2 & 6, Modus ponens ($\Rightarrow$_E)
8. q	from 7, Simplification ($\wedge$_E)
9. r	assumption
10. $q \wedge r$	from 8 & 9, Conjunction ($\wedge$_I)
11. t	from 4 & 11, Modus ponens ($\Rightarrow$_E)
12. $\neg \neg r$	from 9, Law of Negation
13. $\neg w \vee s$	from 3 & 12, Disjunctive syllogism ($\vee$_E)
14. w	from 7, Simplification ($\wedge$_E)
15. $\neg \neg w$	from 14, Law of Negation
16. s	from 13 & 15, Disjunctive syllogism ($\vee$_E)
17. $s \wedge t$	from 16 & 11, Conjunction ($\wedge$_I)
18. $r \Rightarrow s \wedge t$	from 9-17, Conditional proof ($\Rightarrow$_I)
19. $q \wedge (r \Rightarrow s \wedge t)$	from 8 & 18, Conjunction ($\wedge$_I)
20. $p \Rightarrow q \wedge (r \Rightarrow s \wedge t)$	from 5-19, Conditional proof ($\Rightarrow$_I)

4. Use indirect proof to prove the following.

$s \Rightarrow (r \Rightarrow p) ,\ s \Rightarrow (\neg r \Rightarrow p) ,\ \neg q \vee s ,\ q \vee s \ \ |\!-\ \ p$

Answer

1. $s \Rightarrow (r \Rightarrow p)$	premise
2. $s \Rightarrow (\neg r \Rightarrow p)$	premise
3. $\neg q \vee s$	premise
4. $q \vee s$	premise
5. $\neg s$	assumption
6. $\neg q$	from 3 & 5, Disjunctive syllogism ($\vee$_E)
7. q	from 4 & 5, Disjunctive syllogism ($\vee$_E)
5. $\neg \neg s$	from 5-7 (lines 6 & 7), Indirect proof ($\neg$_I)
6. s	from 5, Law of Negation
7. $r \Rightarrow p$	from 1 & 6, Modus ponens ($\Rightarrow$_E)
8. $\neg r \Rightarrow p$	from 2 & 6, Modus ponens ($\Rightarrow$_E)
9. $\neg p$	assumption
10. $\neg r$	from 7 & 9, Modus tollens ($\Rightarrow$_E)
11. $\neg \neg r$	from 8 & 9, Modus tollens ($\Rightarrow$_E)
12. $\neg \neg p$	from 9-11 (lines 10 & 11), Indirect proof ($\neg$_I)
13. p	from 12, Law of Negation

5. For each of the following natural-language arguments, do the following.

- Encode the argument as a propositional-logic inference. Be sure to define the meanings of the prime propositions that you use.
- Use the shorter truth table method to show that the argument is valid.
- Prove that the argument is valid using inference rules and logical equivalences.

a. *If the red button is clicked, the screen shrinks but the image's size doesn't change. The text disappears whenever the screen shrinks, but the submit button is still visible. So, when the red button is clicked, the image doesn't change in size and the submit button remains visible.*

Answer

Let

p = *The red button is clicked*.

q = *The screen shrinks*.

r = *The image's size changes*.

s = *The text disappears*.

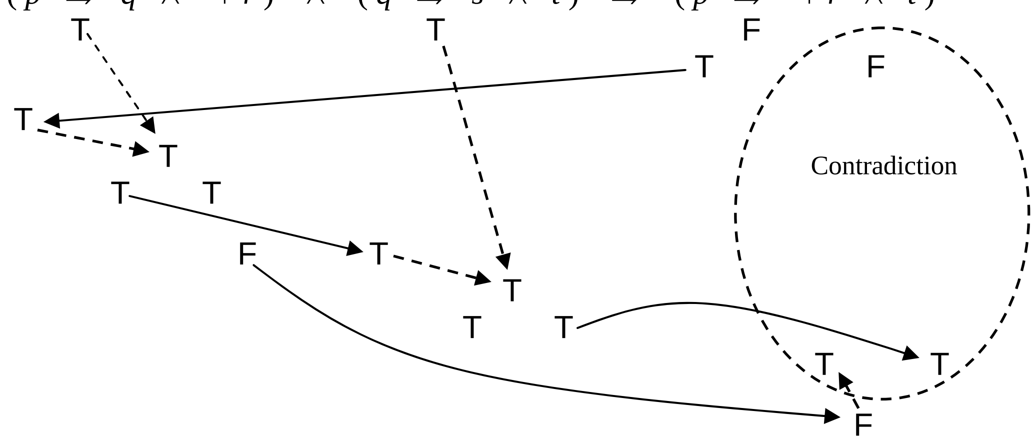
t = *The submit button is visible*.

Then the encoding of the argument is

$$p \Rightarrow q \wedge \neg r\,,\ q \Rightarrow s \wedge t \ \ |\!-\ \ p \Rightarrow \neg r \wedge t$$

The following use of the shorter truth-table method shows this argument valid.

$$(p \Rightarrow q \wedge \neg r) \wedge (q \Rightarrow s \wedge t) \Rightarrow (p \Rightarrow \neg r \wedge t)$$



Since no assignment of truth values to the constituent primes results in this implication being T, it is a tautology. Hence, the corresponding argument is valid,

Answer to **5a** (continued)

The following proves that this argument is valid using inference rules and logical equivalences.

1. $p \Rightarrow q \wedge \neg r$ premise
2. $q \Rightarrow s \wedge t$ premise
   3. p assumption
   4. $q \wedge \neg r$ from 1 & 3, Modus ponens ($\Rightarrow$_E)
   5. q from 4, Simplification ($\wedge$_E)
   6. $\neg r$ from 4, Simplification ($\wedge$_E)
   7. $s \wedge t$ from 2 & 5, Modus ponens ($\Rightarrow$_E)
   8. t from 7, Simplification ($\wedge$_E)
   9. $\neg r \wedge t$ from 6 & 8, Conjunction ($\wedge$_I)
10. $p \Rightarrow \neg r \wedge t$ from 3-9, Conditional proof ($\Rightarrow$_I)

b. *The crosswalk is closed when the train is in the station. The stationmaster is in his office whenever the crosswalk is closed or a train is being monitored. So, the stationmaster is in his office because the gates are closed, and, when the gates are closed, the train is in the station*.

Answer

Let

p = *The crosswalk is closed.*
q = *The train is in the station.*
r = *The stationmaster is in his office.*
s = *A train is being monitored.*
t = *The gates are closed.*

Then the encoding of the argument is

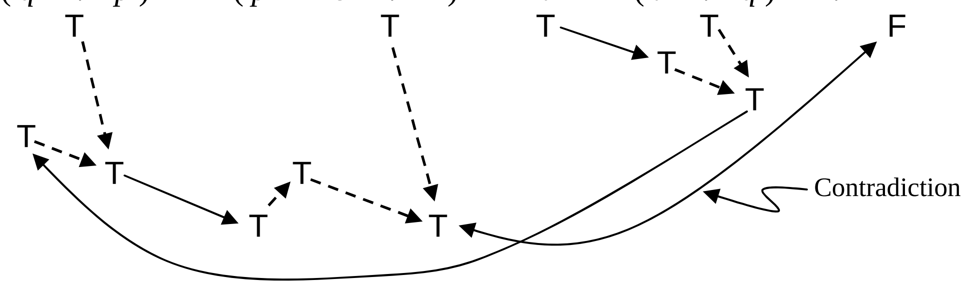
$$q \Rightarrow p\ ,\ p \vee s \Rightarrow r\ ,\ t\ ,\ t \Rightarrow q \quad |\text{-} \quad r$$

You might argue that the last two premising should be conjoined, $t \wedge (t \Rightarrow q)$, but this does not affect what can be deduced from the premises.

Answer to **5b** (continued)

The following use of the shorter truth-table method shows this argument valid.

$$(q \Rightarrow p) \wedge (p \vee s \Rightarrow r) \wedge t \wedge (t \Rightarrow q) \Rightarrow r$$



Since no assignment of truth values to the constituent primes results in this implication being T, it is a tautology. Hence, the corresponding argument is valid,

The following proves that this argument is valid using inference rules and logical equivalences.

1.	$q \Rightarrow p$	premise
2.	$p \vee s \Rightarrow r$	premise
3.	t	premise
4.	$t \Rightarrow q$	premise
5.	q	from 4 & 3, Modus ponens ($\Rightarrow\_E$)
6.	p	from 1 & 5,
7.	$p \vee s$	from 6, Addition ($\vee\_I$)
8.	r	from 2 & 7, Modus ponens ($\Rightarrow\_E$)