COMP 681 Formal Methods Spring 2008 Assignment 3—<u>Solutions</u>

1. Consider the following conditional.

```
if P then  {
  if Q or R then S1
  else S2
}
else if R and T then  {
  if Q then S1
  else S2
}
else S1
```

a. In the language of propositional logic, write the conditions under which each of the statements S_1 and S_3 is executed. Simplify, at least move all ¬'s inward so they apply only to primes and factor out common terms.

b. Rewrite the conditional by filling in the following blanks.

```
if P then  {
  if ________________ then S2
  else S1
}
else if _______________ then S1
else if _____ then S2
else S1
```

Answer

```
if P then  {
  if not Q and not R then S2
  else S1
}
else if not R or not T then S1
else if not Q then S2
else S1
```

2. A group of experts from different departments decides that procedure S_1 is done when P is false, that S_2 is done if R holds, and that S_3 is done Q is false but R is true.

a. Write a GCL alternation statement that captures these requirements. (The conditions won't be mutually exclusive at this point.)

Answer

```
if  ¬ P   →    S1
□    R       →   S2
□  ¬ Q ∧ R   →   S3
fi
```

b. You point out to the experts that their conditions are not mutually exclusive. They decide among themselves that S_3 should take priority over the other two statements and that S_2 should take priority over S_1. Rewrite the conditions so that they are mutually exclusive and respect these priorities.

Answer

The guard for S_3 is unchanged: $\neg Q \wedge R$

The guard for S_2 becomes

$\neg(\neg Q \wedge R) \wedge R$	
$\langle\equiv\rangle\ (\neg\neg Q \vee \neg R) \wedge R$	De Morgan's 1st Law
$\langle\equiv\rangle\ (Q \vee \neg R) \wedge R$	Law of Negation
$\langle\equiv\rangle\ Q \wedge R \vee \neg R \wedge R$	Distributive Law ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ Q \wedge R \vee \mathit{false}$	Law of Contradiction
$\langle\equiv\rangle\ Q \wedge R$	Simplification (3.21)

The guard for S_1 becomes

$\neg R \wedge \neg(\neg Q \wedge R) \wedge \neg P$	
$\langle\equiv\rangle\ \neg R \wedge (\neg\neg Q \wedge \neg R) \wedge \neg P$	De Morgan's 1st Law
$\langle\equiv\rangle\ \neg R \wedge (Q \vee \neg R) \wedge \neg P$	Law of Negation
$\langle\equiv\rangle\ \neg R \wedge (\neg R \vee Q) \wedge \neg P$	Commutativity of $\vee$
$\langle\equiv\rangle\ \neg R \wedge \neg P$	Absorption (3.23)
$\langle\equiv\rangle\ \neg P \wedge \neg R$	Commutativity of $\wedge$

So the new statement is

```
if  ¬ P ∧ ¬ R  →   S1
□  Q ∧ R        →  S2
□  ¬ Q ∧ R     →  S3
fi
```

c. For the cases not covered in **b**, add another alternative whose guard covers the remaining cases yet is disjoint from each of the other three guards. The command associated with this guard should be `skip`.

Answer

Form the conjunction of the negation of the guards and simplify.

$\neg(\neg P \wedge \neg R) \wedge \neg(Q \wedge R) \wedge \neg(\neg Q \wedge R)$	
$\langle\equiv\rangle\ (\neg\neg P \vee \neg\neg R) \wedge (\neg Q \wedge \neg R) \wedge (\neg\neg Q \wedge \neg R)$	De Morgan's 1st Law (3×)
$\langle\equiv\rangle\ (P \vee R) \wedge (\neg Q \wedge \neg R) \wedge (Q \wedge \neg R)$	Law of Negation (3×)
$\langle\equiv\rangle\ (P \vee R) \wedge (\neg Q \wedge Q \vee \neg R)$	Distributive Law ($\vee$ over $\wedge$)
$\langle\equiv\rangle\ (P \vee R) \wedge (\mathit{false} \vee \neg R)$	Law of Contradiction
$\langle\equiv\rangle\ (P \vee R) \wedge \neg R$	Simplification (3.21)
$\langle\equiv\rangle\ P \wedge \neg R \vee R \wedge \neg R$	Distributive Law ($\wedge$ over $\vee$)
$\langle\equiv\rangle\ P \wedge \neg R \vee \mathit{false}$	Law of Contradiction
$\langle\equiv\rangle\ P \wedge \neg R$	Simplification (3.21)

So the final statement is

```
if  ¬ P ∧ ¬ R  →   S1
□  Q ∧ R        →  S2
□  ¬ Q ∧ R     →  S3
□  P ∧ ¬ R     →  skip
fi
```

d. Translate the alternative statement in c into the language used in section 3.3 Minimize the number of primes that must be evaluated.

Answer

```
if  R  then  {
  if  Q  then  S2
  else  S3
}
else if  not P  then  S1
```

3. Convert the following into CNF.

a. $p \wedge \neg q \Leftrightarrow q \Rightarrow p$

Answer

$p \wedge \neg q \Leftrightarrow q \Rightarrow p$

Step 1:

$\langle\equiv\rangle\ (p \wedge \neg q \Rightarrow (q \Rightarrow p)) \wedge ((q \Rightarrow p) \Rightarrow p \wedge \neg q)$	Law of Equivalence

Step 2:

$\langle\equiv\rangle\ (p \wedge \neg q \Rightarrow \neg q \vee p) \wedge (\neg q \vee p \Rightarrow p \wedge \neg q)$	Law of Implication (2×)
$\langle\equiv\rangle\ (\neg (p \wedge \neg q) \vee \neg q \vee p) \wedge (\neg (\neg q \vee p) \vee p \wedge \neg q)$	Law of Implication (2×)

Step 3:

$\langle\equiv\rangle\ (\neg p \vee \neg\neg q \vee \neg q \vee p) \wedge (\neg (\neg q \vee p) \vee p \wedge \neg q)$	De Morgan's 1st Law
$\langle\equiv\rangle\ (\neg p \vee \mathit{true} \vee p) \wedge (\neg (\neg q \vee p) \vee p \wedge \neg q)$	Law of Excluded Middle
$\langle\equiv\rangle\ (\mathit{true} \vee p) \wedge (\neg (\neg q \vee p) \vee p \wedge \neg q)$	Simplification (3.19)
$\langle\equiv\rangle\ \mathit{true} \wedge (\neg (\neg q \vee p) \vee p \wedge \neg q)$	Simplification (3.19)
$\langle\equiv\rangle\ \neg (\neg q \vee p) \vee p \wedge \neg q$	Simplification (3.18)
$\langle\equiv\rangle\ \neg\neg q \wedge \neg p \vee p \wedge \neg q$	De Morgan's 2nd Law
$\langle\equiv\rangle\ q \wedge \neg p \vee p \wedge \neg q$	Law of Negation

Step 4:

$\langle\equiv\rangle\ (q \wedge \neg p \vee p) \wedge (q \wedge \neg p \vee \neg q)$	Distributive Law (∨ over ∧)
$\langle\equiv\rangle\ (q \vee p) \wedge (\neg p \vee p) \wedge (q \wedge \neg p \vee \neg q)$	Distributive Law (∨ over ∧)
$\langle\equiv\rangle\ (q \vee p) \wedge \mathit{true} \wedge (q \wedge \neg p \vee \neg q)$	Law of Excluded Middle
$\langle\equiv\rangle\ (q \vee p) \wedge (q \wedge \neg p \vee \neg q)$	Simplification (3.18)
$\langle\equiv\rangle\ (q \vee p) \wedge (q \vee \neg q) \wedge (\neg p \vee \neg q)$	Distributive Law (∨ over ∧)
$\langle\equiv\rangle\ (q \vee p) \wedge \mathit{true} \wedge (\neg p \vee \neg q)$	Law of Excluded Middle
$\langle\equiv\rangle\ (q \vee p) \wedge (\neg p \vee \neg q)$	Simplification (3.18)

b. $((\neg p \Rightarrow q \vee s) \Rightarrow r) \vee (q \wedge \neg s \Rightarrow \neg p)$

Answer

$((\neg p \Rightarrow q \vee s) \Rightarrow r) \vee (q \wedge \neg s \Rightarrow \neg p)$

Step 1: Nothing to do.

Step 2:

$\langle\equiv\rangle\ (\neg\neg p \vee q \vee s \Rightarrow r) \vee (q \wedge \neg s \Rightarrow \neg p)$	Law of Implication
$\langle\equiv\rangle\ (p \vee q \vee s \Rightarrow r) \vee (q \wedge \neg s \Rightarrow \neg p)$	Law of Negation
$\langle\equiv\rangle\ \neg(p \vee q \vee s) \vee r \vee \neg(q \wedge \neg s) \vee \neg p$	Law of Implication (2×)

Step 3:

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg s \vee r \vee \neg(q \wedge \neg s) \vee \neg p$	De Morgan's 2nd Law
$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg s \vee r \vee \neg q \vee \neg\neg s \vee \neg p$	De Morgan's 1st Law
$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg s \vee r \vee \neg q \vee s \vee \neg p$	Law of Negation
$\langle\equiv\rangle\ \neg p \vee \neg p \wedge \neg q \wedge \neg s \vee r \vee \neg q \vee s$	Commutativity of $\vee$
$\langle\equiv\rangle\ \neg p \vee r \vee \neg q \vee s$	Absorption (3.22)

This is now in CNF—there's only one conjunct. There is nothing to do for Step 4.

4. Convert the CNF wffs you get in problem 3 into the normal form with implications but no negations.

a.

Answer

For the first conjunct:

$q \vee p$

$\langle\equiv\rangle\ false \vee q \vee p$	Simplification (3.21)
$\langle\equiv\rangle\ \neg true \vee q \vee p$	Since true $\langle\equiv\rangle \neg false$
$\langle\equiv\rangle\ true \Rightarrow q \vee p$	Law of Implication

For the second conjunct:

$\neg p \vee \neg q$

$\langle\equiv\rangle\ \neg p \vee \neg q \vee false$	Simplification (3.21)
$\langle\equiv\rangle\ \neg(p \wedge q) \vee false$	De Morgan's 1st Law
$\langle\equiv\rangle\ p \wedge q \Rightarrow false$	Law of Implication

Substituting these two results into the wff in CNF:

$(true \Rightarrow q \vee p) \wedge (p \wedge q \Rightarrow false)$

b.

Answer

There is only one conjunct:

$\neg p \vee r \vee \neg q \vee s$

$\langle\equiv\rangle\ \neg p \vee \neg q \vee r \vee s$	Commutativity of $\vee$
$\langle\equiv\rangle\ \neg(p \wedge q) \vee r \vee s$	De Morgan's 2nd Law
$\langle\equiv\rangle\ p \wedge q \Rightarrow r \vee s$	Law of Implication

This is the final answer.

5. Convert the following into DNF.

$(p \vee q \Rightarrow r) \wedge (r \vee s \Rightarrow p)$

Answer

$(p \vee q \Rightarrow r) \wedge (r \vee s \Rightarrow p)$

Step 1: Nothing to do.

Step 2:

$\langle\equiv\rangle\ (\neg (p \vee q) \vee r) \wedge (\neg (r \vee s) \vee p)$ Law of Implication (2×)

Step 3:

$\langle\equiv\rangle\ (\neg p \wedge \neg q \vee r) \wedge (\neg r \wedge \neg s \vee p)$ De Morgan's 2nd Law (2×)

Step 4:

$\langle\equiv\rangle\ (\neg p \wedge \neg q \vee r) \wedge \neg r \wedge \neg s \ \vee\ (\neg p \wedge \neg q \vee r) \wedge p$ Distrib. Law ($\wedge$ over $\vee$)

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ r \wedge \neg r \wedge \neg s \ \vee\ (\neg p \wedge \neg q \vee r) \wedge p$ Distrib. Law ($\wedge$ over $\vee$)

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ \mathit{false} \wedge \neg s \ \vee\ (\neg p \wedge \neg q \vee r) \wedge p$ Law of Contradiction

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ \mathit{false} \ \vee\ (\neg p \wedge \neg q \vee r) \wedge p$ Simplification (3.20)

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ (\neg p \wedge \neg q \vee r) \wedge p$ Simplification (3.21)

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ \neg p \wedge \neg q \wedge p \ \vee\ r \wedge p$ Distrib. Law ($\wedge$ over $\vee$)

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ p \wedge \neg p \wedge \neg q \ \vee\ r \wedge p$ Commutativity of $\wedge$

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ \mathit{false} \wedge \neg q \ \vee\ r \wedge p$ Law of Contradiction

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ \mathit{false} \ \vee\ r \wedge p$ Simplification (3.20)

$\langle\equiv\rangle\ \neg p \wedge \neg q \wedge \neg r \wedge \neg s \ \vee\ r \wedge p$ Simplification (3.20)

6. Use the technique in the Addendum to Part 3 to convert the following in DNF to CNF. Simplify where possible.

$(p \wedge \neg q) \vee r \vee (r \wedge q)$

Answer

There are three disjuncts. So, in the resulting CNF wff, each conjunct will be a disjunction of three literals.

The first disjunct has two literals to choose from, the second has one, and the third has two. So the number of conjuncts in the resulting CNF formula will be 2×1×2 = 4.

In constructing the CNF wff, we choose a literal from the first disjunct, then one from the second (there is only one here), and, finally, one from the third. We go through all the choices in the third disjunct before choosing a new literal in the second (again, there is only one), and we go through all the literals in the second disjunct before going through the literals in the first.

The answer is

$(p \vee r \vee r) \ \wedge\ (p \vee r \vee q) \ \wedge\ (\neg q \vee r \vee r) \ \wedge\ (\neg q \vee r \vee q)$

$\langle\equiv\rangle\ (p \vee r) \ \wedge\ (p \vee r \vee q) \ \wedge\ (\neg q \vee r) \ \wedge\ (\neg q \vee r \vee q)$ Idempotence of $\vee$ (2×)

$\langle\equiv\rangle\ (p \vee r) \ \wedge\ (p \vee r \vee q) \ \wedge\ (\neg q \vee r) \ \wedge\ (q \vee \neg q \vee r)$ Commutativity of $\vee$

$\langle\equiv\rangle\ (p \vee r) \ \wedge\ (p \vee r \vee q) \ \wedge\ (\neg q \vee r) \ \wedge\ (\mathit{true} \vee r)$ Law of Excluded Middle

$\langle\equiv\rangle\ (p \vee r) \ \wedge\ (p \vee r \vee q) \ \wedge\ (\neg q \vee r) \ \wedge\ \mathit{true}$ Simplification (3.19)

$\langle\equiv\rangle\ (p \vee r) \ \wedge\ (p \vee r \vee q) \ \wedge\ (\neg q \vee r)$ Simplification (3.18)