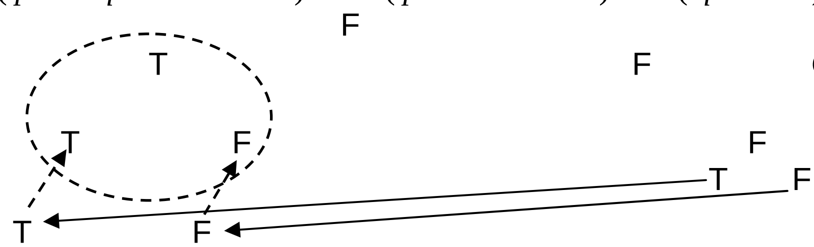
COMP 681 Formal Methods Spring 2008 Assignment 2—<u>Solutions</u>

1. Use the shorter truth table method to determine whether the following are tautologies.

a. $(\, p \vee q \Rightarrow r \wedge s \,) \Rightarrow (\, p \Rightarrow r \vee s \,) \wedge (\, q \Rightarrow r \,)$

Answer

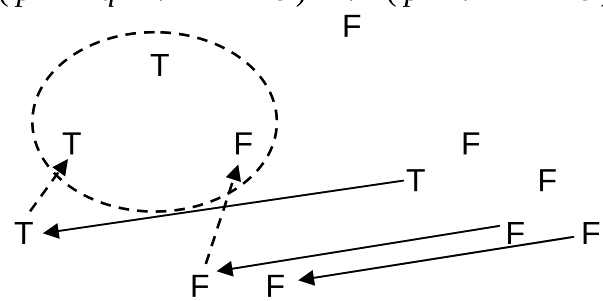
$(\, p \;\vee\; q \;\Rightarrow\; r \;\wedge\; s \,) \;\Rightarrow\; (\, p \;\Rightarrow\; r \;\vee\; s \,) \;\wedge\; (\, q \;\Rightarrow\; r \,)$



Choice: $p \Rightarrow r \vee s$ is F
or $q \Rightarrow r$ is F
<u>Choose</u> $q \Rightarrow r$ F

Having reached a contradiction with this choice, try the other choice.

$(\, p \;\vee\; q \;\Rightarrow\; r \;\wedge\; s \,) \;\Rightarrow\; (\, p \;\Rightarrow\; r \;\vee\; s \,) \;\wedge\; (\, q \;\Rightarrow\; r \,)$



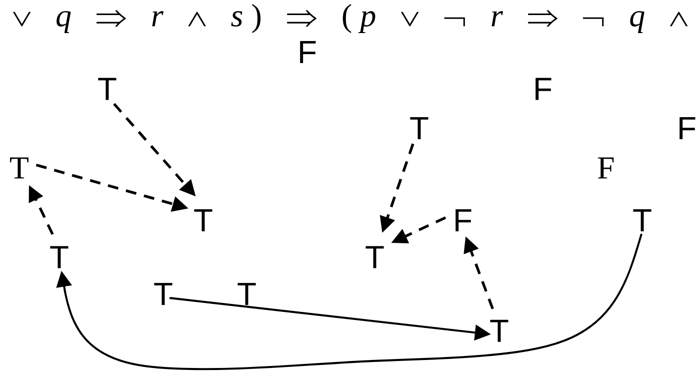
Choice: $p \Rightarrow r \vee s$ is F
or $q \Rightarrow r$ is F
<u>Choose</u> $p \Rightarrow r \vee s$ F

(Note that either F for r or F for s is sufficient for F for $r \wedge s$.)
Since reached a contradiction in both cases, we have shown that there is no consistent assignment of truth values to the primes that results in this formula being F, i.e., it is a tautology.

b. $(\, p \vee q \Rightarrow r \wedge s \,) \Rightarrow (\, p \vee \neg\, r \Rightarrow \neg\, q \wedge s \,)$

Answer

$(\, p \;\vee\; q \;\Rightarrow\; r \;\wedge\; s \,) \;\Rightarrow\; (\, p \;\vee\; \neg\; r \;\Rightarrow\; \neg\; q \;\wedge\; s \,)$



Choice: $\neg\, q$ if F or s is F
<u>Choose</u> $\neg\, q$ F

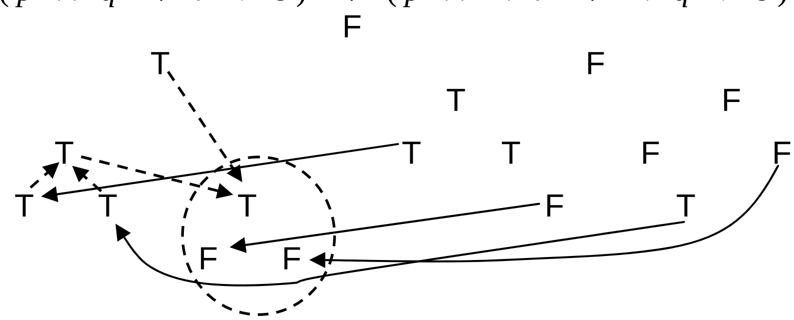
We have now found a consistent assignment of truth values to the primes that makes this formula F (p, q, r, and s are all T). So this formula is not a tautology.

c. $p \wedge q \Rightarrow r \vee s \;\Leftrightarrow\; p \wedge \neg r \Rightarrow \neg q \vee s$

Answer

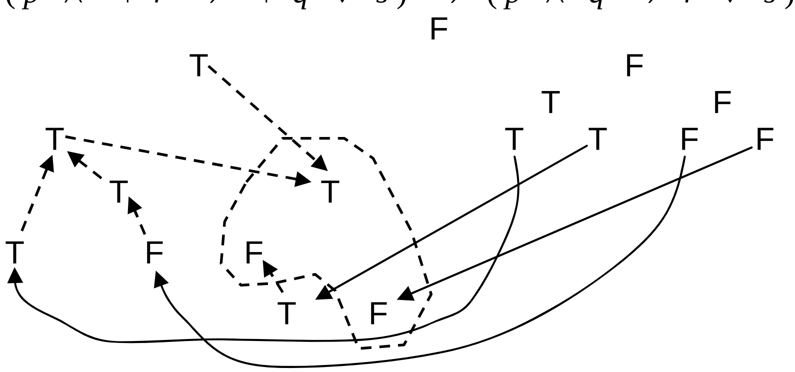
For this biconditional, we check whether the conditional in each direction is a tautology.

$(p \wedge q \Rightarrow r \vee s) \Rightarrow (p \wedge \neg r \Rightarrow \neg q \vee s)$



Having assumed that this implication is F, we reached a contradiction in the assignment of truth values to the primes. We now check the implication in the other direction.

$(p \wedge \neg r \Rightarrow \neg q \vee s) \Rightarrow (p \wedge q \Rightarrow r \vee s)$

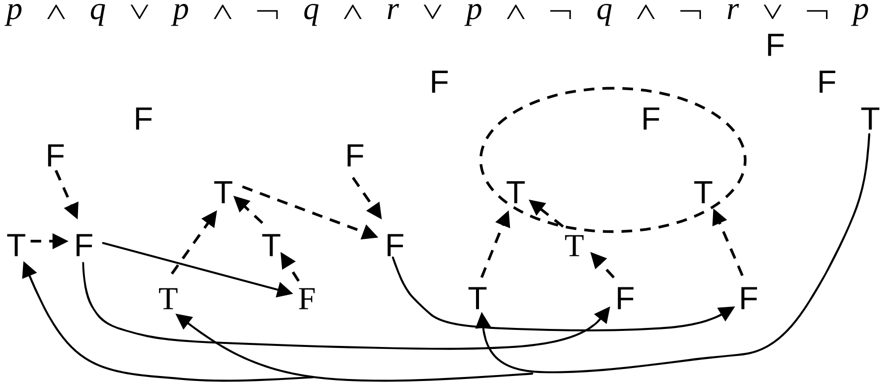


Having assumed that this implication is F, we reached a contradiction in the assignment of truth values to the primes. So the implications in both directions are tautologies, so the biconditional itself is a tautology.

d. $p \wedge q \vee p \wedge \neg q \wedge r \vee p \wedge \neg q \wedge \neg r \vee \neg p$

Answer

$p \;\wedge\; q \;\vee\; p \;\wedge\; \neg\; q \;\wedge\; r \;\vee\; p \;\wedge\; \neg\; q \;\wedge\; \neg\; r \;\vee\; \neg\; p$



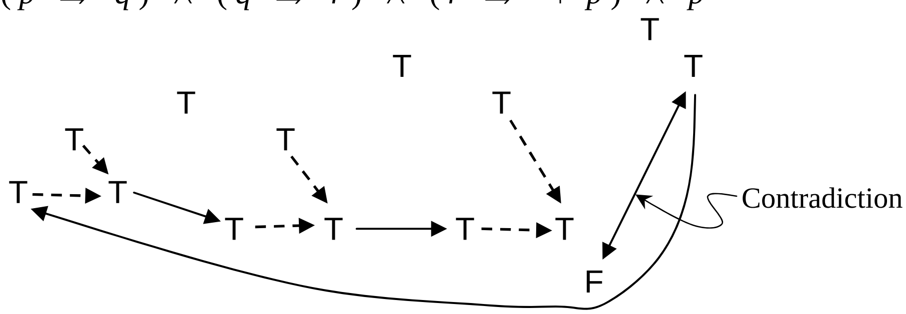
Having assumed that this formula is F, we reached a contradiction in the assignment of truth values to the primes. Thus the formula cannot be F, so it is a tautology.

2. Use the shorter truth table method to determine whether the following are contradictions.

a. $(p \Rightarrow q) \wedge (q \Rightarrow r) \wedge (r \Rightarrow \neg p) \wedge p$

Answer

$(p \;\Rightarrow\; q) \;\wedge\; (q \;\Rightarrow\; r) \;\wedge\; (r \;\Rightarrow\; \neg\; p) \;\wedge\; p$

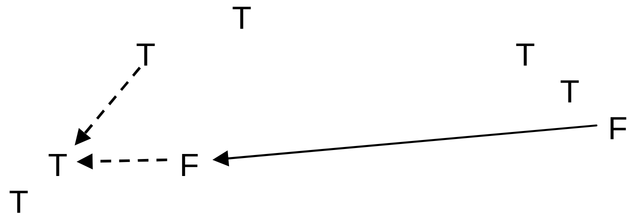


Having assumed that this formula is T, we reached a contradiction in the assignment of truth values to the primes. Thus the formula cannot be T, so it is a contradiction.

b. $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Answer

$(p \;\vee\; q \;\vee\; r) \;\wedge\; (\neg\; p \;\vee\; \neg\; q \;\vee\; \neg\; r)$



Choice: $\neg p \vee \neg q$ is T or $\neg r$ is T
Choose : $\neg r$ T

Choice : p is T or q is T
Choose: p is T

At this point, all required truth values are accounted for, and q can be either T or F. So one assignment that makes this formula T is: r is F, p is T, and q is T (or F). Since there is at least one assignment that makes the formula T, it is not a contradiction.
In fact, this formula is made T by any assignment that makes at least one of p, q, and r T and at least one of them F.

3. Prove the following using a transformational proof. Justify each step. Show all substitutions used to derive appropriate instances of the rules.

a. $\neg p \wedge \neg q \Rightarrow \neg r \;\langle\equiv\rangle\; r \Rightarrow p \vee q$

Answer

$\neg p \wedge \neg q \Rightarrow \neg r$

$\langle\equiv\rangle\; \neg\neg r \;\Rightarrow\; \neg (\neg p \wedge \neg q)$	Contrapositive Law $\{ p / \neg p \wedge \neg q,\; q / \neg r \}$
$\langle\equiv\rangle\; r \;\Rightarrow\; \neg (\neg p \wedge \neg q)$	Law of Negation $\{ p / r \}$
$\langle\equiv\rangle\; r \;\Rightarrow\; \neg\neg p \vee \neg\neg q$	De Morgan's 1st Law $\{ p / \neg p,\; q / \neg q \}$
$\langle\equiv\rangle\; r \;\Rightarrow\; p \vee \neg\neg q$	Law of Negation
$\langle\equiv\rangle\; r \;\Rightarrow\; p \vee q$	Law of Negation $\{ p / q \}$

b. $p \Rightarrow (q \Rightarrow r) \;\langle\equiv\rangle\; p \wedge q \Rightarrow r$

Answer

$p \Rightarrow (q \Rightarrow r)$

$\langle\equiv\rangle\; \neg p \vee (q \Rightarrow r)$	Law of Implication $\{ q / q \Rightarrow r \}$
$\langle\equiv\rangle\; \neg p \vee (\neg q \vee r)$	Law of Implication $\{ p / q,\; q / r \}$
$\langle\equiv\rangle\; (\neg p \vee \neg q) \vee r$	Associative Law for $\vee$ $\{ p / \neg p,\; q / \neg q \}$
$\langle\equiv\rangle\; \neg (p \wedge q) \vee r$	De Morgan's 1st Law
$\langle\equiv\rangle\; p \wedge q \;\Rightarrow\; r$	Law of Implication $\{ p / p \wedge q,\; q / r \}$

c. $p \Rightarrow q \;\langle\equiv\rangle\; \neg (p \wedge \neg q)$

Answer

$p \Rightarrow q$

$\langle\equiv\rangle\; \neg p \vee q$	Law of Implication
$\langle\equiv\rangle\; \neg\neg (\neg p \vee q)$	Law of Negation $\{ p / \neg p \vee q \}$
$\langle\equiv\rangle\; \neg (\neg\neg p \wedge \neg q)$	De Morgan's 2nd Law $\{ p / \neg p \}$
$\langle\equiv\rangle\; \neg (p \wedge \neg q)$	Law of Negation

4. Prove the following using a transformational proof. Justify each step, but you need not show the substitution used to derive the appropriate instance of the laws. You may assume implicit associativity and commutativity of $\wedge$ and $\vee$, and you may use generalized forms of the laws.

a. $(p \Rightarrow q) \Rightarrow (r \Rightarrow s) \ \langle\equiv\rangle\ r \Rightarrow (\neg p \Rightarrow s) \wedge (q \Rightarrow s)$

Answer

$(p \Rightarrow q) \Rightarrow (r \Rightarrow s)$

$\langle\equiv\rangle\ \neg(r \Rightarrow s) \Rightarrow \neg(p \Rightarrow q)$	Contrapositive Law
$\langle\equiv\rangle\ \neg\neg(r \Rightarrow s) \vee \neg(p \Rightarrow q)$	Law of Implication
$\langle\equiv\rangle\ (r \Rightarrow s) \vee \neg(p \Rightarrow q)$	Law of Negation
$\langle\equiv\rangle\ (r \Rightarrow s) \vee \neg(\neg p \vee q)$	Law of Implication
$\langle\equiv\rangle\ (r \Rightarrow s) \vee \neg\neg p \wedge \neg q$	De Morgan's 1st Law
$\langle\equiv\rangle\ (r \Rightarrow s) \vee p \wedge \neg q$	Law of Negation
$\langle\equiv\rangle\ (\neg r \vee s) \vee p \wedge \neg q$	Law of Implication
$\langle\equiv\rangle\ \neg r \vee (s \vee p \wedge \neg q)$	Associativity of $\vee$
$\langle\equiv\rangle\ \neg r \vee (s \vee p) \wedge (s \vee \neg q)$	Distributive Law ($\vee$ over $\wedge$)
$\langle\equiv\rangle\ \neg r \vee (p \vee s) \wedge (s \vee \neg q)$	Commutativity of $\vee$
$\langle\equiv\rangle\ \neg r \vee (\neg\neg p \vee s) \wedge (s \vee \neg q)$	Law of Negation
$\langle\equiv\rangle\ \neg r \vee (\neg p \Rightarrow s) \wedge (s \vee \neg q)$	Law of Implication
$\langle\equiv\rangle\ \neg r \vee (\neg p \Rightarrow s) \wedge (\neg q \vee s)$	Commutativity of $\vee$
$\langle\equiv\rangle\ \neg r \vee (\neg p \Rightarrow s) \wedge (q \Rightarrow s)$	Law of Implication
$\langle\equiv\rangle\ r \Rightarrow (\neg p \Rightarrow s) \wedge (q \Rightarrow s)$	Law of Implication

b. $p \wedge q \wedge \neg r \Rightarrow r \vee \neg(p \wedge q) \ \langle\equiv\rangle\ p \Rightarrow (q \Rightarrow r)$

Answer

$p \wedge q \wedge \neg r \Rightarrow r \vee \neg(p \wedge q)$

$\langle\equiv\rangle\ \neg(p \wedge q \wedge \neg r) \vee r \vee \neg(p \wedge q)$	Law of Implication
$\langle\equiv\rangle\ \neg(p \wedge q) \vee \neg\neg r \vee r \vee \neg(p \wedge q)$	De Morgan's 2nd Law
$\langle\equiv\rangle\ \neg(p \wedge q) \vee r \vee r \vee \neg(p \wedge q)$	Law of Negation
$\langle\equiv\rangle\ \neg(p \wedge q) \vee r \vee \neg(p \wedge q)$	Idempotence of $\vee$
$\langle\equiv\rangle\ \neg(p \wedge q) \vee \neg(p \wedge q) \vee r$	Commutativity of $\vee$
$\langle\equiv\rangle\ \neg(p \wedge q) \vee r$	Idempotence of $\vee$
$\langle\equiv\rangle\ (\neg p \vee \neg q) \vee r$	De Morgan's 1st Law
$\langle\equiv\rangle\ \neg p \vee (\neg q \vee r)$	Associativity of $\vee$
$\langle\equiv\rangle\ \neg p \vee (q \Rightarrow r)$	Law of Implication
$\langle\equiv\rangle\ p \Rightarrow (q \Rightarrow r)$	Law of Implication