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A Nonlinear Pricing Model with Lump-Sum Transportation Costs

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Abstract

In this article we introduce a nonlinear pricing model with lump-sum transportation costs. In contrast to previously-used models that include per-unit costs, we find that all consumers who connect to the market will purchase the same amount. Our results are shown to be consistent with pricing policies observed in some markets.

I. Introduction

Nearly all analysis that deals with how firms and consumers respond to spatial differentiation has assumed that consumers face a per-unit shipping cost which is strictly increasing in both the quantity of goods purchased and the distance to the outlet. This is normally the case for firms buying larger quantities of an input.[1]

In 1981 Spulber derived the optimal nonlinear pricing scheme for a monopolist whose customers face such a per-unit shipping charge. However, in many purchasing situations consumers incur a cost of traveling to an outlet, but no per-unit shipping costs. In these situations, it is necessary to determine how consumers react to price changes and additional transportation costs. This understanding will allow us to determine how a firm

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[1] For analyses of per-unit shipping costs, see for example Greenthut and Ohta (1979), Furlong and Slotsve (1983), Coyte and Lindsey (1988).

should set prices when the transportation cost increases with distance, but not with quantity.[2]

The distinction between the two types of costs may be seen in the choices modern consumers face via markets on the internet. Purchasing compact discs over the internet will normally carry some shipping cost which is increasing in quantity. However, the same customer shopping at a "brick and mortar" store faces no such marginal shipping cost, only the fixed cost of traveling to the store. Therefore, internet customers react to these charges as an increase in price, whereas brick and mortar customers react to the cost of going to the store as a fixed access fee.[3]

In contrast to Spulber, we derive the optimal nonlinear pricing scheme for a monopolist in a market facing lump-sum transportation costs, comparing the outcome under a linear monopoly price (the same price for each unit sold). We explore this behavior in the models that follow, examining the impact of nonlinear pricing on the firm and consumers.

In the next section we describe the model, and follow in section III with the general results of the model. In section IV we extend our results with explicit solutions for some simple functional forms (that were also used by Spulber) for demand and the distribution of consumer locations. This is followed by the conclusion and we discuss our results in the context of warehouse clubs in the appendix.

II. Lump Sum Transport Cost Model

We consider the problem of a monopolist selecting an optimal policy when its customers are spatially dispersed. The monopolist sells a good $x \in R^{+}$ and selects the marginal price schedule for customers. The monopolist is located at 0 on a real line.

Customers are distributed according to the cumulative distribution *G(r)* where $r \in [0,\infty)$, and $G(0)=0$, $g(r) \equiv dG(r)/dr$ for every $r \in R$. The consumers are assumed to be identical in all respects other than location. The consumer faces exogenous transportation costs *C(r)* which are increasing in *r* but not in quantity. The consumer will maximize a utility function of *x* and a unit-priced numeraire, *y*: $U(x,y)=u(x)+y$ where *u* is

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[2] Stahl (1982) is one of very few cases where lump-sum transportation costs are considered.

[3] For example, see Oi (1971) and Ng and Weisser (1974) for discussion of two-part tariff models. A two-part tariff involves an access fee plus per-unit charges, similar to a county fair or warehouse club.

twice differentiable, with $\partial u/\partial x > 0, \partial^2 u/\partial x^2 < 0$. Assuming away income effects, we can write the consumer's objective function as:

$$(1) \qquad \max_x [u(x) - P(x) - C(r)]$$

where *P(x)* represents the *total* purchase price of x units. Faced with fixed transportation costs, the consumer will choose to purchase from the monopolist only if his consumers' surplus[4] (*CS*) is at least as large as the lump-sum transportation cost. For simplicity we assume that the lump-sum travel cost faced by a consumer located at *r* is *Cr*, with $C \geq 0$.

We can state the consumer's decision problem as:

Purchase $x^* \; s.t. \; u'(x) = P'(x)$ if $CS = u(x) - P(x) \geq Cr$, 0 otherwise.

Lump-sum travel costs have no impact on the quantity purchased by a consumer, but have the effect of causing some consumers to disconnect from the market if the cost is too high. The travel cost will determine, for a given price, the point on the real line at which disconnection occurs:

$$(2) \qquad r^o = \frac{u(x^*) - P(x^*)}{C}$$

In this paper we consider two specific cases for $P(x)$. First, we will consider the case where the same price is charged for each unit, which we will call the linear price. Then, we will consider the case where a nonlinear pricing function can be used.

III. Main Results

Assuming a constant marginal cost of production for the monopolist (*k>0*), the monopolist will choose the profit maximizing price or nonlinear pricing schedule. In this section we derive the optimal pricing strategies for both linear and nonlinear pricing schemes. In Proposition 1 we consider the linear price, followed by the nonlinear case in Proposition 2.

Proposition 1: Using the linear pricing function $P(x) = px$, the profit maximizing monopoly price satisfies:

[4] Consumers' surplus is the difference between the maximum willingness to pay for a given quantity of a product and the price paid.

(3) $$p = k + \frac{x^*(p)G(r^o(p))}{g(r^o(p))\frac{x^*(p)}{C}x^*(p) - \frac{\partial x^*}{\partial p}G(r^o(p))} > k$$

Proof: The monopolist will choose p to maximize the profit function:

(4) $$\max_p (p-k)\int_0^{r^o(p)} x^*(p)g(r)dr = \max_p (p-k)G(r^o(p))x^*(p)$$

(5) $$\frac{\partial}{\partial p} = x^*(p)G(r^o(p)) + (p-k)\frac{\partial G}{\partial r^o}\frac{\partial r^o}{\partial p}x^*(p) + (p-k)\frac{\partial x^*}{\partial p}G(r^o(p)) = 0$$

Rearranging (5) and using equation (2) gives (3). The right hand side of (3) is positive, because all arguments in the function are positive with the exception of $\partial x^* / \partial p$, which is negative but preceded by a (-) sign. Therefore, $p > k$.

Remark 1

Proposition 1 states that the price will reflect a markup above marginal cost (k) which is increasing in C and $G(r^o)$ but decreasing in $g(r^o)$. In other words, as the number of consumers who are buying is higher, the monopolist can afford to raise the price; however, as the number living near the disconnection point increases, the monopolist can gain many more customers with a slight lowering of the price. This relationship can be rewritten into a familiar profit-maximization condition in economics which relates marginal cost to price and the *price elasticity of demand*[5]*:*

(6) $$\text{k} = p\left(1 + \frac{1}{\left(\frac{p}{x^*(p)G(r^o(p))}\right)\left[\frac{\partial x^*}{\partial p}G(r^o(p)) + \frac{\partial G}{\partial r^o}\frac{\partial r^o}{\partial p}x^*(p)\right]}\right)$$

$$= p(1 + \left(\frac{1}{\frac{dx}{dp}\frac{p}{x}}\right)) = p(1 + \frac{1}{\varepsilon_p})$$

[5] The price elasticity of demand is a measure of the responsiveness of quantity demanded to price, $\varepsilon_p = \frac{\partial x}{\partial p}\frac{p}{x}$ or roughly, the proportionate change in quantity divided by the proportionate change in price.

For linear monopoly pricing, profit maximization entails finding the price such that the marginal cost has the relationship with the price and ε_p, the shown in equation (6). The denominator of (6) is simply the marginal effect on sales from changes in price. The first term in brackets in the denominator is the "inframarginal" effect of price, reflecting the decrease in the quantity purchased by those who connect to the market. The second bracketed term in the denominator is the "extramarginal" effect, reflecting the change in sales caused by changes in the number of customers who connect with the market.

Next we consider a nonlinear pricing function P(x) and prove the following proposition.

Proposition 2: The optimal nonlinear price for the monopolist satisfies the equation:

(7) $$P(x^*) = kx^* + \frac{G(r^o(P))C}{g(r^o(P))},$$

where x^* is the optimal quantity purchased by all consumers who connect to the market, and $P(x)$ is the total price charged to each consumer.

Proof: With a lump-sum travel cost and identical demand functions at each point along the real line, all consumers up to the disconnection point will purchase the same amount ($x^*\ s.t.\ u'(x) = P'(x)$), and all whose travel cost is greater than their consumer surplus will purchase zero. Given this, the monopolist's problem is to choose a nonlinear pricing schedule that will affect x^* for all who connect, and also affect the number who connect through r^o in equation (2). This problem is intuitively similar to the linear pricing problem above. For any consumer who does connect, given a total price $P(x)$, consumers will maximize:

(8) $$\max_x [u(x) - Cr - P(x)], \text{ where}$$

(9) $$u'(x) = P'(x)$$

Let $p(x) \equiv dP(x)/dx$. The monopolist will maximize

(10) $$\max_{P(x)} \int_0^{r^o} \int_0^{x^*} (p(x) - k) g(r) dx dr$$

The problem can be restated in a much simpler framework. Given that all consumers who connect will purchase the same quantity x^*, the monopolist can first determine the profit-maximizing quantity, and then determine the profit-maximizing total price, $P(x^*)$ based on the amount of disconnection which will occur.

In this context we can show that the monopolist will want to choose x^* for each consumer who purchases where

(11) $u'(x^*) = P'(x^*) = k$.

By way of contradiction, suppose that the profit maximizing $\hat{x}$ was s.t. $u'(\hat{x}) > k$, implying that $\hat{x}$ <x* because $\frac{\partial u}{\partial x} > 0, \frac{\partial^2 u}{\partial x^2} < 0$. The profit for each consumer connecting will be $P(\hat{x}) - k\hat{x}$. However, the consumer and producer would both gain if the producer sold a larger package, $\hat{x} + \varepsilon$, and charged a total price of $P(\hat{x}) + u'(x+\varepsilon)\varepsilon$ because $u'(x+\varepsilon)\varepsilon > k\varepsilon$, increasing profits. Therefore, $\hat{x}$ as described is not the profit-maximizing quantity.

On the other hand, suppose that $\hat{x}$ is s.t. $u'(\hat{x}) = P'(\hat{x}) < k$. The profit for each consumer connecting will be $P(\hat{x}) - k\hat{x}$. However, higher profit can be made with a bundle $\hat{x} - \varepsilon$, and charging a total price of $P(\hat{x}) - u'(x+\varepsilon)\varepsilon$ because the revenue loss is smaller than the reduction in costs ($u'(x+\varepsilon)\varepsilon < k\varepsilon$). Therefore $\hat{x}$ is not the profit maximizing quantity

Hence, we see that the profit-maximizing quantity is $x=x^*$ which satisfies $u'(x^*) = P'(x^*) = k$.

Given this x^*, the monopolist only needs to choose the total price, P, to maximize profits. The choice of $P(x^*)$ will affect the number of consumers who connect to the market. From (2) we have $r^o = \frac{u(x^*) - P(x^*)}{C}$. Thus, the monopolist's maximization problem is

(12) $$\max_P (P - kx^*) \int_0^{r^o(P)} g(r)dr = \max_P (P - kx^*) G(r^o(P))$$

Differentiating (12) w.r.t. P gives the first-order condition:

(13) $$\frac{\partial}{\partial P} \Rightarrow \int_0^{r^o(p)} g(r)dr - (P - kx^*) g(r^o) \tfrac{1}{C} = 0$$
$$\Rightarrow G(r^o) - (P - kx^*) g(r^o) \tfrac{1}{C} = 0$$

which implicitly defines the profit maximizing *P* as:

$$(14) \qquad P(x^*) = kx^* + \frac{G(r^o(P))C}{g(r^o(P))}$$

This concludes the proof of Proposition 2.

Remark 2

Equation (12) is similar to (3) in that the profit-maximizing price increases with *C* and $G(r^o)$ but is decreasing in $g(r^o)$. This total price could be implemented with any nonlinear pricing schedule which would induce the consumers to purchase *x** as described above. One such function would be to charge an admission fee or other fixed charge equal to $G(r^o(P))C / g(r^o(P))$, and then charge *k* per unit purchased. Or, the monopolist could simply charge a package price for *x** units. In any case, such a marginal pricing schedule must charge *k* (marginal cost) for the last unit purchased.

Now we compare the results under the two pricing arrangements. Let x_l^* and x_{nl}^* denote the quantity purchased by each consumer under the linear and nonlinear pricing arrangements, respectively.

Proposition 3: For each consumer that connects, $x_{nl}^* \geq x_l^*$.

Proof : Suppose that $x_{nl}^* < x_l^*$. Since $u'(x)$ is a decreasing function we have $u'(x_{nl}^*) > u'(x_l^*)$. From Proposition 2 we know that the nonlinear price sets the quantity purchased s.t. $u'(x_{nl}^*) = k$ for the last unit purchased. Also customers under the linear price will purchase an amount s.t. $u'(x_l^*) = p$. Hence $k = u'(x_{nl}^*) > u'(x_l^*) = p$ which is a contradiction to Proposition 1, which states that the profit-maximizing linear price $p > k$.

Remark 3

Proposition 3 illustrates the key benefit of nonlinear pricing, which is to induce consumers to purchase a larger quantity without having to reduce the price of all units sold in order to do so. This is the reason why nonlinear pricing will almost always increase profits.[6]

It is difficult to take the analysis further in a general model. The pricing and other results of interest are similar, but cannot be directly compared without many more restrictions on the functional forms. In the following section we derive additional results using some simple explicit forms for utility and distribution of consumer locations.

IV. Analysis with Explicit Functional Forms

In order to derive more results, and for comparison of results with models involving per unit transportation costs, we will now assume some common explicit functional forms:[7]

(15) $$u(x) = \alpha x - (1/2)\beta x^2, \alpha > 0, \beta > 0$$

(16) $$G(r) = ar^b, a > 0, b > 0$$

With these specific functional forms, the optimal linear price can be solved from:

(17) $$\max_p (p-k)G(r^o(p))x^*(p)$$

Using equation (15), the demand function ($x_l^*(p)$) satisfies $\max_x [u(x) - px]$, so that

(18) $$x_l^*(p) = \frac{\alpha - p}{\beta}$$

Hence, $\partial x / \partial p = -1/\beta$. Combining (15), (18), and (2) we infer

(19) $$r_l^o = \frac{(\alpha - p)^2}{2C\beta}$$

By using Proposition 1 and combining (16), (18), and (19) we obtain the linear price

(20) $$p_l = \frac{k(2b+1) + \alpha}{2(b+1)}$$

For the explicit functional forms the quantity sold per customer in the linear case is

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[6] Greenhut and Greenhut(1977) show that for per-unit transportation costs, only a narrow class of utility functions will yield no improvement with nonlinear pricing.
[7] As used in Spulber (1981).

(21) $$x_l^* = \frac{\alpha - k}{\beta} \frac{2b+1}{2b+2}$$

where we have used (18) and (20). On the other hand, for the nonlinear quantity we have that $u'(x_{nl}^*) = k$ (see (11)) which together with (15) yields

(22) $$x_{nl}^* = \frac{\alpha - k}{\beta}$$

Obviously (22)>(21), which for these explicit function forms demonstrates a special case of Proposition 3.

To derive the nonlinear price for explicit functional forms we use (16) in Proposition 2 and obtain

(23) $$P(x^*) = kx^* + \frac{rC}{b}$$

Combining (2), (22) and (23), implies

(24) $$P_{nl} = \frac{(\alpha - k)(\alpha + 2bk + k)}{2\beta(1+b)}$$

Proposition 4: The average price charged per unit is the same for the linear price and the nonlinear pricing scheme.

Proof: For the explicit solutions we have

$$p_l = \frac{k(2b+1)+\alpha}{2(b+1)} \text{ and } P_{nl} = \frac{(\alpha - k)(\alpha + 2bk + k)}{2\beta(1+b)}$$

$$\text{Average nonlinear price} = \frac{P_{nl}}{x^*} = \frac{\dfrac{(\alpha - k)(\alpha + 2bk + k)}{\beta 2(b+1)}}{\dfrac{\alpha - k}{\beta}} = \frac{k(2b+1)+\alpha}{2(b+1)} = p_l\,,$$

which concludes the proof.

Remark 4: While a nonlinear pricing scheme would charge higher prices for earlier units and k for the last unit purchased by each consumer, on average they are equal. However, as shown in Proposition 3, the monopolist induces consumers to purchase more with the nonlinear pricing scheme.

Proposition 5: The market area served is larger under the linear pricing scheme, i.e. the following holds: $r_l^o > r_{nl}^o$.

Proof: From (19) and (20) we see that

$$r_l^o = \frac{(\alpha - k)^2(2b+1)^2}{8\beta C(b+1)^2}. \quad (25)$$

On the other hand, from (23) we have

$$r_{nl}^o = \frac{(P(x_{nl}^*) - kx_{nl}^*)b}{C} \quad (26)$$

which together with (22) and (24) yield

$$r_{nl}^o = \frac{b(\alpha - k)^2}{2\beta C(b+1)} \quad (27)$$

It can easily be seen by comparing (25) and (27) that $r_l^o > r_{nl}^o$. Thus the market area is larger for the linear pricing case, which concludes the proof of Proposition 5.

V. Conclusion

We have described the profit maximizing behavior that can be seen in a monopolized market when the transportation costs are lump-sum, as in most consumer markets. In Proposition 2 we found that the optimal nonlinear price in such a market could be charged as a two-part tariff, i.e. a fixed access fee plus marginal cost pricing for all units purchased. However, in Proposition 4 we saw that for one general class of utility and consumer density functions, the average price per unit purchased will be the same.

Though the average price is the same, by choosing the nonlinear pricing strategy the sales will increase per customer, but the number of customers will decrease. Overall, however, this results in higher profits for the seller.

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Appendix: Application to Membership Warehouse Clubs

Let us apply these results to the warehouse club industry in the United States. These clubs charge a fixed yearly fee for membership, then sell their goods at near cost to their members. Table 1 shows recent results for the three major warehouse clubs in the U.S.; Sam’s Club, COSTCO, and BJ’s. The financial reporting date is shown in parentheses.

For these three retailers, we see that the Net Sales (revenue from goods sold) is very close to the Total Cost of Sales. We also see that the operating income is made up almost entirely by the fixed (membership) fees, very much like our nonlinear pricing monopolist in Proposition 2.

These club retailers could charge a simple linear price for their goods, but would not make as much profit if they did so. Occasionally these retailers will allow some nonmembers to shop at the store (as a way of advertising the store), but in these cases the nonmember must pay the posted price for the items purchased plus some percentage.

In reality these stores not only face consumers who differ in location, but the consumers will also differ in many other respects. There are undoubtedly some types of consumers who would prefer not to pay a membership fee in order to enter the store, particularly those who will purchase smaller volumes of goods. These customers will likely pay higher prices per unit, but buy less than those who decide to join the warehouse-type stores. Wal-Mart Corporation (owner of Sam’s Club) provides both retail options to consumers through different types of stores, and thus can profit from this market segmentation.

Table 1	**BJ's (May 1,2004)**		**CostCo (Aug.31, 2003)**		**Sam's (Apr. 30, 2004 Q1)**	
	(in $1,000s)		(in 1,000,000s)		(in 1,000,000s)	
Net Sales	1,610,958		41,693		8,374	
Cost of Sales		1,497,600		37,235		*
Selling, general, and administrative expenses		123,200		4,097		*
Total cost of sales	1,620,800		41,332		*	
Membership fees	33,100		853		210	
Operating Income	26,605		1,157		267	

*Sam's Club is part of Wal-Mart Corporation, and does not separately report all costs.
Data obtained from financial disclosure reports.