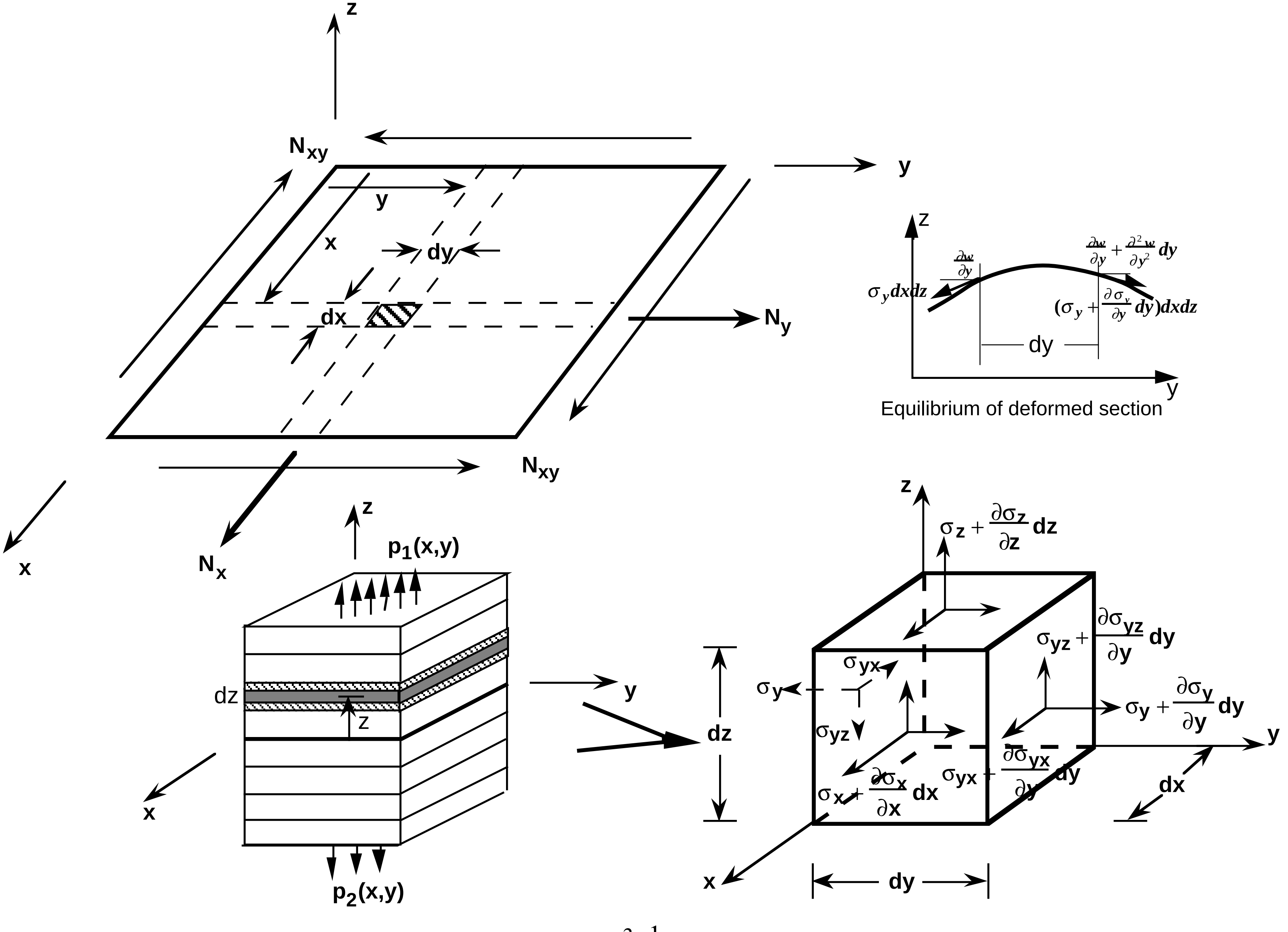
5.8.1 Governing Equations of Equilibrium of Composite laminates



Applied Loading:

1. Traction load p(x,y,t)

2. Body force: Fx, Fy, and Fz; Includes Inertial, Magnetic, etc. $\boxed{\rho\ddot{u},\ \rho\ddot{v},\ \text{and}\ \rho\ddot{w}}$

3.1 Equations of Equilibrium

Consider the equilibrium of the elemental volume dx x dy x dz in y-direction

$$\left(\sigma_y+\frac{\partial\sigma_y}{\partial y}dy-\sigma_y\right)dxdz+\left(\sigma_{zy}+\frac{\partial\sigma_{zy}}{\partial z}dz-\sigma_{zy}\right)dxdy+\left(\sigma_{xy}+\frac{\partial\sigma_{xy}}{\partial x}dx-\sigma_{xy}\right)dydz+F_y dxdydz=0$$

Dividing throughout by dx.dy.dz, we get

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}+\frac{\partial\sigma_{xy}}{\partial x}+F_y=0 \qquad (2)$$

Similarly, the equilibrium in x-direction is

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+F_x=0 \qquad (1)$$

A general equilibrium in z-direction includes the transverse force components due to in-plane stresses. These terms are a result of equilibrium of the deformed body and are important for buckling and dynamic stability problems. Consider the equilibrium in z-direction,

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+F_z+\sigma_x\frac{\partial^2 w}{\partial x^2}+2\sigma_{xy}\frac{\partial^2 w}{\partial x\partial y}+\sigma_y\frac{\partial^2 w}{\partial y^2}=0 \qquad (3)$$

3.2 Equations of Equilibrium in terms of Force Resultants

Integrate equations 1 to 3 with respect to z. For a laminate having 'n' number of layers,

$$\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\frac{\partial\sigma_x}{\partial x}\right)_k dz+\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\frac{\partial\sigma_{yx}}{\partial y}\right)_k dz+\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\frac{\partial\sigma_{zx}}{\partial z}\right)_k dz+\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(F_x\right)_k dz=0$$

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}+\overline{F}_x=0 \qquad (4)$$

Where, Nx, and Nyx are Inplane stress (Force resultants) and $\overline{F}_x$ is the average body force per unit volume. Similarly, force resultant expression in y-direction is

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}+\overline{F}_y=0 \qquad (5)$$

Integration of Equation 3, yields

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p_1-p_2+\overline{F}_z+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}=0 \qquad (6)$$

Where p_1 and p_2 are the pressure loading on faces z=+h/2 and -h/2. In laminated plate theory, we can replace the two pressures, $\mathbf{p_1 - p_2 = p}$, resultant pressure loading..

Moment Resultants:

$$\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\frac{\partial \sigma_x}{\partial x}\right)_k z dz + \sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\frac{\partial \sigma_{yx}}{\partial y}\right)_k z dz + \sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\frac{\partial \sigma_{zx}}{\partial z}\right)_k z dz = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0 \tag{7}$$

Similarly,

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{8}$$

Substituting Eqs. 7 and 8 in 6 results in a fundamental equations of motion of a laminated plate.

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p + \overline{F}_z + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{9}$$

Equations 1, 2, and 9 are general governing equations of motion of a laminated plate. These equation is valid for static, buckling, and dynamic (free vibration, dynamic response, and dynamic stability) problems. Depending on the type of problem, the equations simplify accordingly.

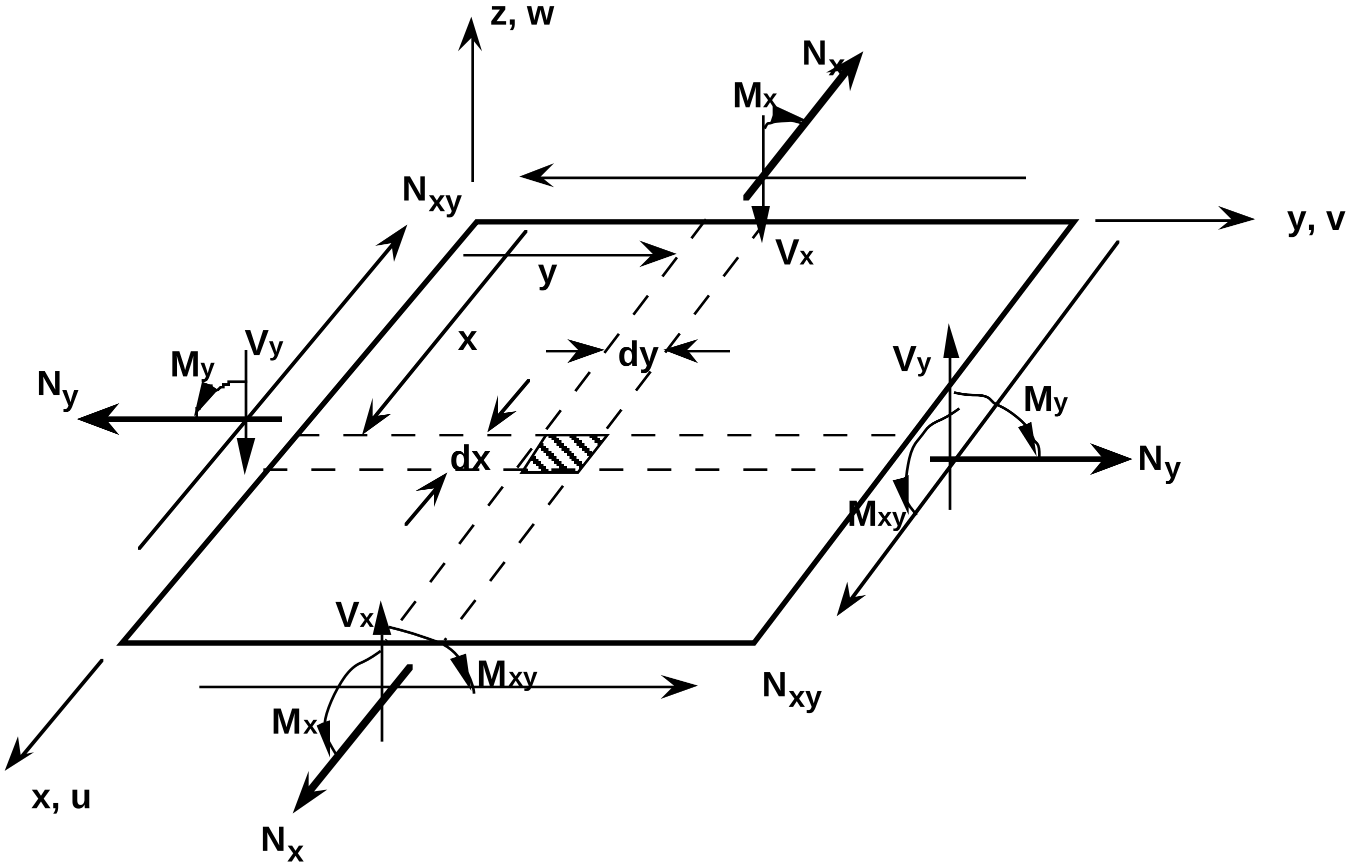
Static Stress Analysis

$$\frac{\partial \boldsymbol{N_x}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{N_{yx}}}{\partial \boldsymbol{y}} = 0$$

$$\frac{\partial \boldsymbol{N_{xy}}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{N_y}}{\partial \boldsymbol{y}} = 0$$

$$\frac{\partial^2 \boldsymbol{M_x}}{\partial \boldsymbol{x}^2} + 2\frac{\partial^2 \boldsymbol{M_{xy}}}{\partial \boldsymbol{x}\partial \boldsymbol{y}} + \frac{\partial^2 \boldsymbol{M_y}}{\partial \boldsymbol{y}^2} + \boldsymbol{p} + \overline{\boldsymbol{F}}_{\boldsymbol{z}} = 0$$

Static Analysis



Governing equations of equilibrium

$N_{x,x} + N_{yx,y} = 0$ (1) $N_{xy,x} + N_{y,y} = 0$ (2)

$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} = -p$ (3)

Substituting for force & moment results, we get for no thermal and moisture loading

$$\begin{aligned}&A_{11}u_{,xx}+2A_{16}u_{,xy}+A_{66}u_{,yy}+A_{16}v_{,xx}+\left(A_{12}+A_{66}\right)v_{,xy}+A_{26}v_{,yy}\\&-B_{11}w_{,xxx}-3B_{16}w_{,xxy}-\left(B_{12}+2B_{66}\right)w_{,xyy}-B_{26}w_{,yyy}=0\end{aligned}\tag{4}$$

$$\begin{aligned}&A_{16}u_{,xx}+\left(A_{12}+A_{66}\right)u_{,xy}+A_{26}u_{,yy}+A_{66}v_{,xx}+2A_{16}v_{,xy}+A_{22}v_{,yy}\\&-B_{16}w_{,xxx}-\left(B_{12}+2B_{66}\right)w_{,xxy}-3B_{26}w_{,xyy}-B_{22}w_{,yyy}=0\end{aligned}\tag{5}$$

$$\begin{aligned}&D_{11}w_{,xxxx}+4D_{16}w_{,xxxy}+2\left(D_{12}+2D_{66}\right)w_{,xxyy}+4D_{26}w_{,xyyy}+D_{22}w_{,yyyy}\\&-B_{11}u_{,xxx}-3B_{16}u_{,xxy}-\left(B_{12}+2B_{66}\right)u_{,xyy}-B_{26}u_{,yyy}\\&-B_{16}v_{,xxx}-\left(B_{12}+2B_{66}\right)v_{,xxy}-3B_{26}v_{,xyy}-B_{22}v_{,yyy}=p\end{aligned}\tag{6}$$

Equations 4 to 6 represent the general governing differential equations of equilibrium of a statically loaded anisotropic laminate. The equations involve coupled differentials of u, v, and w displacements. The equations significantly simplifies for a symmetric laminate (Bij = 0).

Equations of Equilibrium for Symmetric laminates:

$$A_{11}u_{,xx}+2A_{16}u_{,xy}+A_{66}u_{,yy}+A_{16}v_{,xx}+\left(A_{12}+A_{66}\right)v_{,xy}+A_{26}v_{,yy}=0\tag{7}$$

$$A_{16}u_{,xx}+\left(A_{12}+A_{66}\right)u_{,xy}+A_{26}u_{,yy}+A_{66}v_{,xx}+2A_{16}v_{,xy}+A_{22}v_{,yy}=0\tag{8}$$

$$D_{11}w_{,xxxx}+4D_{16}w_{,xxxy}+2\left(D_{12}+2D_{66}\right)w_{,xxyy}+4D_{26}w_{,xyyy}+D_{22}w_{,yyyy}=p\tag{9}$$

Kirchhoff Shear (V)

Moment equilibrium equation:

$$M_{x,x} + M_{xy,y} - Q_x = 0$$

$$M_{x,x} - V_x = 0 \quad \text{or} \quad V_x = Q_x - M_{xy,y} \tag{10}$$

Vx is called the Kirchhoff shear force.

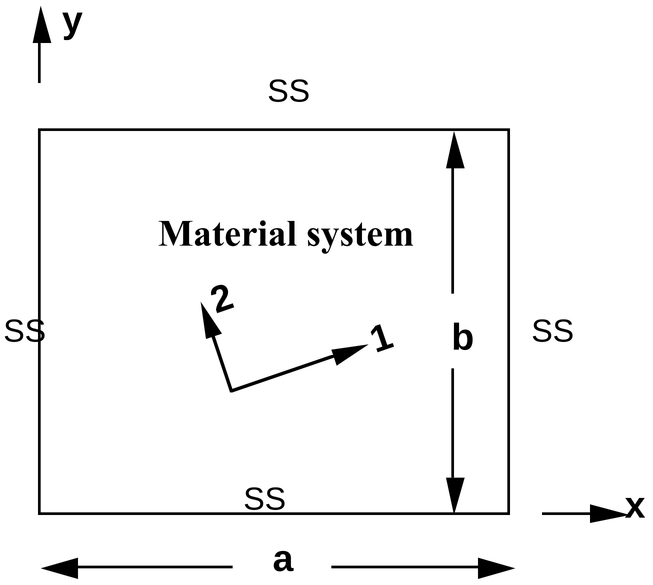
5.2 Boundary Conditions

General Curved Boundary Condition:

Analysis Methods:

- Classical Methods
- Energy Methods
- Numerical Methods (Finite Element and Boundary Element Methods)

Analysis of Simply-Supported Laminated Plate Subjected to an Uniform Load (p)



Case 1: Specially Orthotropic Laminate
"Material axes parallel to the Plate axes" - All coupling (Normal-Shear and Bending-Stretching) terms are ZERO.

GDE: $D_{11}w_{,xxxx} + 2\left(D_{12} + 2D_{66}\right)w_{,xxyy} + D_{22}w_{,yyyy} = p$ (1)

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = -D_{11}\, w_{,xx} - D_{12}\, w_{,yy} = 0$

@ y=0 & b: $w = 0$ & $M_y = -D_{12}\, w_{,xx} - D_{22}\, w_{,yy} = 0$

Selection of Displacement Functions: Because the GDE & Bcs are even derivatives of x and y, we can select a solution in the form:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{2}$$

Loading:

Different types of loading can be expressed using the Fourier series, as

$$p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{3}$$

The Fourier coefficients are calculated for each type of loading as follows

$$\int_0^a \int_0^b p(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy = \int_0^a \int_0^b \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy$$

$$p_{mn} = \frac{4}{ab} \int_0^a \int_0^b p(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy \tag{4}$$

For an uniform loading of qo,

$$p_{mn} = \frac{4q_o}{ab} \int_0^a \int_0^b \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy = \frac{16q_o}{\pi^2 mn}, \qquad \text{For odd numbers of m \& n.}$$

Solution:

Substituting Eqs 2 & 3 in Eq 1 (GDE), we get

$$a_{mn} = \frac{p_{mn}}{\pi^4\left\{D_{11}\left(\frac{m}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{a}\right)^4\right\}} \tag{5}$$

$$\therefore w = \sum_{m=1,3}^{\infty}\sum_{n=1,3}^{\infty}\frac{16q_o \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\pi^6\left\{D_{11}\left(\frac{m}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{a}\right)^4\right\}} \tag{6}$$