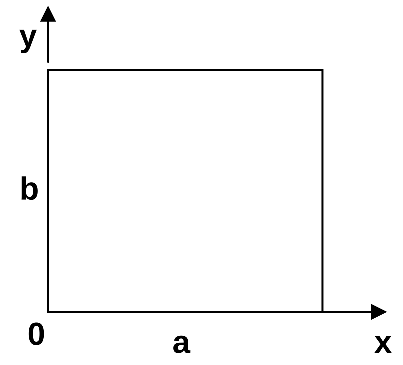
Analysis of Simply-Supported Symmetric Angle-Ply Laminate under UDL Using:
a) Raleigh Ritz Total Potential Energy (TPE) Method.
b) Analogy with Skewed Isotropic Plate.

a) ***Raleigh Ritz Total Potential Energy (TPE) Method:***

- Deflection of S-S plate can be expressed as in the following series summation:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$



- The Governing Differential Equation of Equilibrium is,

$$D_{11}w,_{xxxx} + 4D_{16}w,_{xxxy} + 2(D_{12} + 2D_{66})w,_{xxyy} + 4D_{26}w,_{xyyy} + D_{22}w,_{yyyy} = P$$

- The simply supported edge boundary conditions are given as,

$$x = 0,a : w = 0, M_X = -D_{11}w,_{xx} - D_{12}w,_{yy} - 2D_{16}w,_{xy} = 0$$

$$x = 0,b : w = 0, M_Y = -D_{12}w,_{xx} - D_{22}w,_{yy} - 2D_{26}w,_{xy} = 0$$

- Solution form for Raleigh-Ritz Total Potential Energy (TPE) equation is,

$$\Pi = \frac{1}{2}\iint [\, D_{11}(w,_{xx})^2 + 4D_{16}w,_{xx}w,_{xy} + 2D_{12}w,_{xx}w,_{yy} + 4D_{66}(w,_{xy})^2 + 4D_{26}w,_{yy}w,_{xy} + D_{22}(w,_{yy})^2 - 2pw \,]\, dxdy$$

- Differentiating ***w*** with respect to ***xx, xy*** *and* ***yy*** the following expressions were obtained,

$$w_{,xx} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\left(\frac{m\pi}{a}\right)^2 (-1)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$w_{,xy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\left(\frac{mn}{ab}\right)\pi^2 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$w_{,yy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\left(\frac{n\pi}{b}\right)^2 (-1)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Substituting $w_{,xx}$, $w_{,yy}$ *and* $w_{,xy}$ in TPE and performing integration will give the following:

$$D_{11} \Rightarrow \iint (w_{,xx})^2 \, dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\pi^4}{4} \frac{m^4 b}{a^3} a_{mn}^2$$

$$D_{16} \Rightarrow \iint w_{,xy}\, w_{,xx} \, dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2 \frac{m^3 n}{a^3 b} \pi^4 \frac{a}{2m\pi}\left(1-cos\,2m\pi\right)\frac{b}{2n\pi}\left(1-cos\,2n\pi\right)=0$$

$$D_{12} \Rightarrow \iint w_{,yy}\, w_{,xx} \, dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\pi^4}{4} \frac{m^2 n^2}{ab} a_{mn}^2$$

$$D_{66} \Rightarrow \iint (w,_{xy})^2 dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\pi^4}{4} \frac{m^2 n^2}{ab} a_{mn}^2$$

$$D_{26} \Rightarrow \iint w,_{xy} w,_{yy} dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2 \frac{n^3 m}{b^3 a} \pi^4 \frac{a}{2m\pi}(1-\cos 2m\pi)\frac{b}{2n\pi}(1-\cos 2n\pi) = 0$$

$$D_{22} \Rightarrow \iint (w,_{yy})^2 dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\pi^4}{4} \frac{n^4 a}{b^3} a_{mn}^2$$

$$p \Rightarrow \iint w dxdy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \frac{ab}{mn\pi^2}(1-\cos m\pi)(1-\cos n\pi)$$

$$= p\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\left(\frac{ab}{mn\pi^2}\right)\left[1-(-1)^m-(-1)^n+(-1)^{m+n}\right]$$

By substituting all integration results in TPE and minimizing Π with respect to a_{mn},

$$a_{mn} = \frac{4a^4 P\left[1-(-1)^m\right]\left[1-(-1)^n\right]}{mn\pi^6\left[m^4 D_{11} + 2m^2 n^2 (D_{12} + 2D_{66}) + n^4 D_{22}\right]}$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4a^4 P\left[1-(-1)^m\right]\left[1-(-1)^n\right]}{mn\pi^6\left[(m^4 D_{11} + 2m^2 n^2 (D_{12} + 2D_{66}) + n^4 D_{22}\right]} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}$$

$$\frac{D_{11}}{D_{22}} = 1 \quad , \quad \frac{(D_{12} + 2D_{66})}{D_{11}} = 1.5 \quad , \quad \frac{D_{16}}{D_{11}} = \frac{D_{26}}{D_{11}} = -0.5$$

- for m=n =1 $W_{max} = 0.003329\frac{pa^4}{D_{11}}$ for m=n =3 $W_{max} = 0.003231\frac{pa^4}{D_{11}}$
- for m=n =5 $W_{max} = 0.003239\frac{pa^4}{D_{11}}$ for m=n =7 $W_{max} = 0.0032378\frac{pa^4}{D_{11}}$
- for m=n=9 $W_{max} = 0.0032382\frac{pa^4}{D_{11}}$ for m=n=11 $W_{max} = 0.0032381\frac{pa^4}{D_{11}}$

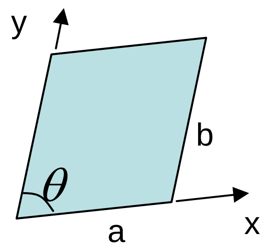
b) ***Analogy with Skewed Isotropic Plate:***

For skewed isotropic plate, the governing differential equation according to Ashton* is:

$$D\left[w,_{xxxx} - 4\alpha\cos(\theta)w,_{xxxy} + 2\alpha^2\left(1+2\cos^2(\theta)\right)w,_{xxyy} - 4\alpha^3\cos(\theta)w,_{xyyy} + \alpha^4 w,_{yyyy}\right] = a^4\sin^4(\theta)P$$

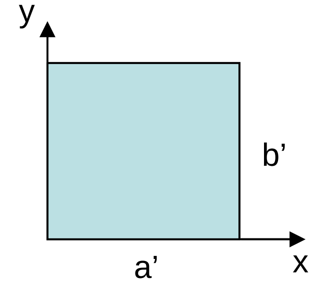
θ : Skew angle.

α : aspect ratio $= \dfrac{a}{b}$



For symmetric angle-ply square laminate, the governing differential equation is:

$$D_{11}w,_{xxxx} + 4D_{16}w,_{xxxy} + 2(D_{12}+2D_{66})w,_{xxyy} + 4D_{26}w,_{xyyy} + D_{22}w,_{yyyy} = P'$$



$$D_{11}\left[w,_{xxxx} + 4\frac{D_{16}}{D_{11}}w,_{xxxy} + 2\frac{(D_{12}+2D_{66})}{D_{11}}w,_{xxyy} + 4\frac{D_{26}}{D_{11}}w,_{xyyy} + \frac{D_{22}}{D_{11}}w,_{yyyy}\right] = P'$$

$$\frac{D_{11}}{a'^4}\left[a'^4\frac{\partial^4 w}{\partial x^4} + 4\frac{D_{16}}{D_{11}}a'^4\frac{\partial^4 w}{\partial x^3\partial y} + 2\frac{(D_{12}+2D_{66})}{D_{11}}a'^4\frac{\partial^4 w}{\partial x^2\partial y^2} + 4\frac{D_{26}}{D_{11}}a'^4\frac{\partial^4 w}{\partial x\,\partial y^3} + \frac{D_{22}}{D_{11}}a'^4\frac{\partial^4 w}{\partial y^4}\right] = P'$$

* J. E. Ashton, "An Analogy for Certain anisotropic Plates.", *J. Composite Materials*, Vol. 3, 1969, pp. 355-358.

$$\frac{D_{11}}{a'^4}\left(\frac{\partial^4 w}{\partial\left(\frac{x}{a'}\right)^4}+4\frac{D_{16}}{D_{11}}\frac{a'}{b'}\frac{\partial^4 w}{\partial\left(\frac{x}{a'}\right)^3\partial\left(\frac{y}{b'}\right)}+2\frac{(D_{12}+2D_{66})}{D_{11}}\left(\frac{a'}{b'}\right)^2\frac{\partial^4 w}{\partial\left(\frac{x}{a'}\right)^2\partial\left(\frac{y}{b'}\right)^2}+4\frac{D_{26}}{D_{11}}\left(\frac{a'}{b'}\right)^3\frac{\partial^4 w}{\partial\left(\frac{x}{a'}\right)\partial\left(\frac{y}{b'}\right)^3}+\frac{D_{22}}{D_{11}}\left(\frac{a'}{b'}\right)^4\frac{\partial^4 w}{\partial\left(\frac{y}{b'}\right)^4}\right)=P'$$

$$D_{11}\left[w,_{xxxx}+4\frac{D_{16}}{D_{11}}\alpha' w,_{xxxy}+2\frac{(D_{12}+2D_{66})}{D_{11}}\alpha'^2 w,_{xxyy}+4\frac{D_{26}}{D_{11}}\alpha'^3 w,_{xyyy}+\frac{D_{22}}{D_{11}}\alpha'^4 w,_{yyyy}\right]=a'^4P'$$

For this specific problem it is assumed that α and α' are both equal to 1.

Comparing the two equations:

$$D\left[w,_{xxxx}-4cos(\theta)w,_{xxxy}+2\left(1+2cos^2(\theta)\right)w,_{xxyy}-4cos(\theta)w,_{xyyy}+w,_{yyyy}\right]=a^4 sin^4(\theta)P$$

$$D_{11}\left[W,xxxx+4\frac{D_{16}}{D_{11}}W,xxxy+2\frac{(D_{12}+2D_{66})}{D_{11}}W,xxyy+4\frac{D_{26}}{D_{11}}W,xyyy+\frac{D_{22}}{D_{11}}W,yyyy\right]=a'^4P'$$

$$D_{11}=D$$

$$\frac{D_{16}}{D_{11}}=\frac{D_{26}}{D_{11}}=-cos(\theta)$$

$$\frac{D_{22}}{D_{11}}=1$$

$$\frac{D_{12}+2D_{66}}{D_{11}}=1+2cos^2(\theta)$$

$$a'^4P'=a^4 sin^4(\theta)P$$

To satisfy the same values of the flexural stiffness ratios used in the direct TPE approach (part a), the skew angle has to be 60*o*.

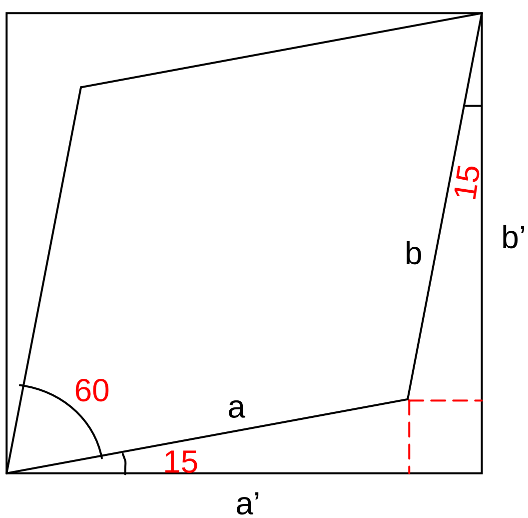
$$\frac{D_{16}}{D_{11}} = \frac{D_{26}}{D_{11}} = -cos(60) = -0.5$$

$$\frac{D_{22}}{D_{11}} = 1$$

$$\frac{D_{12} + 2D_{66}}{D_{11}} = 1 + 2\,cos^2(60) = 1.5$$

$$P' = \frac{a^4}{a'^4} sin^4(60) P = \frac{9}{16}\frac{a^4}{a'^4} P$$

The following sketch and analysis are valid for the special case when the aspect ratio for both geometries equals to 1. ($\alpha = \alpha' = 1$)



$$a' = a\,cos(15) + a\,sin(15)$$

$$a' = a(cos(15) + sin(15))$$

$$\left(\frac{a}{a'}\right)^4 = \frac{1}{[cos(15) + sin(15)]^4} = 0.44444$$

$$P' = \frac{P}{4}$$

Boundary Conditions:

Simply-supported antisymmetric angle-ply:

$$@\, x = 0 \,\&\, a' : w = 0 \,\&\, M_x = -D_{11} w,_{xx} - D_{12} w,_{yy} - 2D_{16} w,_{xy} = 0$$

$$@\, y = 0 \,\&\, b' : w = 0 \,\&\, M_y = -D_{12} w,_{xx} - D_{22} w,_{yy} - 2D_{26} w,_{xy} = 0$$

$But,$

$$@\, x = 0 \,\&\, a' : w,_{yy} = 0$$

$$@\, y = 0 \,\&\, b' : w,_{xx} = 0$$

$Therefore,$

$$-D_{11} w,_{xx} - 2D_{16} w,_{xy} = 0$$

$$\Rightarrow w,_{xx} = -2\frac{D_{16}}{D_{11}} w,_{xy}$$

$$-D_{22} w,_{yy} - 2D_{26} w,_{xy} = 0$$

$$\Rightarrow w,_{yy} = -2\frac{D_{26}}{D_{22}} w,_{xy}$$

But

$$\frac{D_{16}}{D_{11}} = \frac{D_{26}}{D_{22}} = -0.5$$

$$w,_{xx} = w,_{xy} \qquad @\, x = 0, a'$$

$$w,_{yy} = w,_{xy} \qquad @\, y = 0, b'$$

Recall the governing equation of the square antisymmetric laminate.

$$D_{11}\left[W_{,xxxx}+4\frac{D_{16}}{D_{11}}W_{,xxxy}+2\frac{(D_{12}+2D_{66})}{D_{11}}W_{,xxyy}+4\frac{D_{26}}{D_{11}}W_{,xyyy}+\frac{D_{22}}{D_{11}}W_{,yyyy}\right]=a'^4P'$$

As shown in boundary conditions, $w_{,xy}$ can be expressed in terms of $w_{,xx}$ and $w_{,yy}$ in the above equation.

$$W_{,xxxx}+4\frac{D_{16}}{D_{11}}W_{,xxxx}+2\frac{(D_{12}+2D_{66})}{D_{11}}W_{,xxyy}+4\frac{D_{26}}{D_{11}}W_{,yyyy}+\frac{D_{22}}{D_{11}}W_{,yyyy}=\frac{a'^4P'}{D_{11}}$$

But

$$\frac{D_{16}}{D_{11}}=\frac{D_{26}}{D_{11}}=-0.5,\quad \frac{D_{11}}{D_{22}}=1$$

$$\frac{(D_{12}+2D_{66})}{D_{11}}=1.5$$

$$W_{,xxxx}-2W_{,xxxx}+3W_{,xxyy}-2W_{,yyyy}+W_{,yyyy}=\frac{a'^4P'}{D_{11}}$$

or

$$-W_{,xxxx}+3W_{,xxyy}-W_{,yyyy}=\frac{a'^4P'}{D_{11}}$$

From skew plate analogy:

$$P' = \frac{P}{4}$$

$$-w,xxxx + 3w,xxyy - w,yyyy = \frac{a'^4 P}{4D_{11}}$$

By using the same technique as in the specially orthotropic case, an exact solution can be found for the above equation. Where,

$$w = \sum_{m=1,3,..}^{\infty} \sum_{n=1,3,..}^{\infty} a_{mn} \sin\frac{m\pi x}{a'} \sin\frac{n\pi y}{b'}$$

$$w,_{xxxx} = \sum_{m=1,3,..}^{\infty} \sum_{n=1,3,..}^{\infty} m^4 \pi^4 a_{mn} \sin\left(\frac{m\pi x}{a'}\right) \sin\left(\frac{n\pi y}{b'}\right)$$

$$w,_{xxyy} = \sum_{m=1,3,..}^{\infty} \sum_{n=1,3,..}^{\infty} m^2 n^2 \pi^4 a_{mn} \sin\left(\frac{m\pi x}{a'}\right) \sin\left(\frac{n\pi y}{b'}\right)$$

$$w,_{yyyy} = \sum_{m=1,3,..}^{\infty} \sum_{n=1,3,..}^{\infty} n^4 \pi^4 a_{mn} \sin\left(\frac{m\pi x}{a'}\right) \sin\left(\frac{n\pi y}{b'}\right)$$

$$\frac{a'^4 P}{4D_{11}} = \sum_{m=1,3,..}^{\infty} \sum_{1,3,..}^{\infty} P_{mn} \sin\left(\frac{m\pi x}{a'}\right) \sin\left(\frac{n\pi y}{b'}\right)$$

$$P_{mn} = \frac{16\left(\frac{a'^4 P}{4D_{11}}\right)}{\pi^2 mn} = \frac{4a'^4 P}{\pi^2 mnD_{11}}$$

Substituting all derivatives and P_{mn} in the above governing equation,

$$a_{mn} = \frac{4Pa'^4}{\pi^6 mnD_{11}\left(-m^4 + 3m^2n^2 - n^4\right)}$$

$$w(x,y) = \sum_{m=1,3,..}^{\infty}\sum_{n=1,3,..}^{\infty} \frac{4Pa'^4}{\pi^6 mnD_{11}\left(-m^4 + 3m^2n^2 - n^4\right)} sin\left(\frac{m\pi x}{a'}\right) sin\left(\frac{n\pi y}{b'}\right)$$

$$w_{max} = \sum_{m=1,3,..}^{\infty}\sum_{n=1,3,..}^{\infty} \frac{4Pa'^4}{\pi^6 mnD_{11}\left(-m^4 + 3m^2n^2 - n^4\right)} sin\left(\frac{m\pi}{2}\right) sin\left(\frac{n\pi}{2}\right)$$

$$w_{max} = \frac{a'^4 P}{D_{11}} \sum_{m=1,3,..}^{\infty}\sum_{n=1,3,..}^{\infty} \frac{4}{\pi^6 mn\left(-m^4 + 3m^2n^2 - n^4\right)} sin\left(\frac{m\pi}{2}\right) sin\left(\frac{n\pi}{2}\right)$$

$$w_{max} = \beta \frac{a'^4 P}{D_{11}}$$

Fortran program was written to perform the double summation series to obtain the value of β.

FORTRAN PROGRAM

```
sum=0.0
k=11
pi=4.*atan(1.)
   do n=1,k,2
   do m=1,k,2
      Beta=((4./(m*n*pi**6))*sin(m*pi/2.)*sin(n*pi/2.))/(-m**4+3*m**2*n**2-n**4)
      sum=sum+Beta
   enddo
   enddo
print*,  ' Beta=',sum
end
```

By running the FORTRAN program, β converges to 0.004231.

$$w_{max} = 0.004232 \frac{a'^4 P}{D_{11}}$$

The same method solution was presented by *J. E. Ashton* in 1969 as:

$$w_{max} = 0.00425 \frac{a'^4 P}{D_{11}}$$

Table of Summary

TPE method (Orthotropic solution)	Analogy with skewed isotropic plate	Analogy with skewed isotropic plate (*by: Ashton*)
$w_{max} = 0.0032381 \frac{pa^4}{D_{11}}$ m = n = 11	$w_{max} = 0.004232 \frac{a'^4 P}{D_{11}}$ m = n = 11	$w_{max} = 0.00425 \frac{a'^4 P}{D_{11}}$

Comparison between results presented by *Ashton* and current analysis results at different skew angles.

Skew Angle θ	Orthotropic solution* β	Ashton solution* β	Current study β
90	0.00406	0.00406	0.004062
80	0.00394	0.00408	0.004008
63	0.00336	0.00422	0.004229
60	0.00324	0.00425	0.004232
54	0.00301	0.00430	0.004188

* J. E. Ashton, “An Analogy for Certain anisotropic Plates.”, *J. Composite Materials*, Vol. 3, 1969, pp. 355-358.