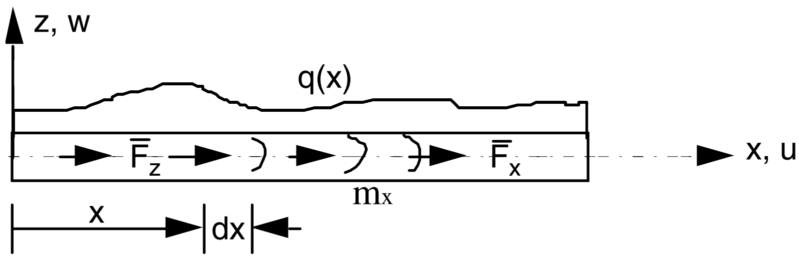
Chapter 4: Analysis of General Anisotropic Laminated Beams

4.1 Governing differential equations.



Loading on the beam:

q(x) - Transverse loading per unit length.

$\bar{F}_x(x)$ - Axial body force + loading per unit length.

$\bar{F}_z(x)$ - Transverse body force per unit length.

$m_x(x)$ - Applied moment along the x-axis.

Equations of equilibrium in terms of Force and Moment resultants:
Note that all the stress resultants are due to mechanical loading, thermal and moisture effects are introduced through CLT equations.

$$N_{x,x} + \bar{F}_x = 0 \qquad (1)$$

$$Q_{x,x} + q + \bar{F}_z = 0 \qquad (2)$$

$$M_{x,x} - Q_x + m_{x,x} = 0 \qquad (3)$$

Combining Eqs 2 & 3, we get

$$M_{x,xx} + m_{x,xx} + q + \bar{F}_z = 0 \qquad (4)$$

Now we will introduce the stress resultant constitutive equations:

$$N_x + N_x^{TH} = A_x \varepsilon_x + B_x \kappa_x = A_x u,_x + B_x w,_{xx} \qquad (5)$$

$$M_x + M_x^{TH} = B_x \varepsilon_x + D_x \kappa_x = B_x u,_x + D_x w,_{xx} \qquad (6)$$

Substituting Eqs 5 & 6 in to Eqs 1 & 4, respectively, and then solving for displacements u and w, we get.

$$w,_{xxxx} = \frac{1}{\overline{D}_x}(q + m_{x,xx} + \overline{F}_z - M^{TH}_{x,xx}) - \frac{1}{\overline{B}_x}(\overline{F}_{x,x} - N^{TH}_{x,xx})$$
$$u,_{xxx} = \frac{1}{\overline{B}_x}(q + m_{x,xx} + \overline{F}_z - M^{TH}_{x,xx}) - \frac{1}{\overline{A}_x}(\overline{F}_{x,x} - N^{TH}_{x,xx}) \qquad (7)$$

<u>Governing Differential Equations for Various Types of Problems</u>

1. Static problems with no body forces.

$$w,_{xxxx} = \frac{1}{\overline{D}_x}(q - M^{TH}_{x,xx}) + \frac{1}{\overline{B}_x} N^{TH}_{x,xx}$$
$$u,_{xxx} = \frac{1}{\overline{B}_x}(q - M^{TH}_{x,xx}) + \frac{1}{\overline{A}_x} N^{TH}_{x,xx} \qquad (8)$$

General solution:

$$w(x) = \frac{1}{\overline{D}_x}\iiint\!\!\int (q - M^{TH}_{x,xx})dx + \frac{1}{\overline{B}_x}\iiint\!\!\int N^{TH}_{x,xx}\,dx + c_1x^3 + c_2x^2 + c_3x + c_4$$
$$u(x) = \frac{1}{\overline{B}_x}\iiint (q - M^{TH}_{x,xx})dx + \frac{1}{\overline{A}_x}\iiint N^{TH}_{x,xx}dx + c_5x^2 + c_6x + c_7 \qquad (9)$$

Thermal and moisture axial force and moment resultants are for zero for no thermal and moisture loading. The seven constants in equation 9 can be determined from the end conditions of the beam.

2. Dynamic Problems

Include time dependent loading, inertial forces, and moment due to any compression loading.

(a) Inertial forces $\overline{F}_z = -\rho A \frac{d^2w}{dt^2}$

(b) Dynamic loading q(x,t)

(c) Compression induced bending moment, m(x) = P w(x)

$$m,_x = P\frac{dw}{dx} \qquad \text{and} \qquad m,_{xx} = P\frac{d^2w}{dx^2}$$

One can include axial inertial effects, however these frequencies are very high compare to the transverse frequencies, hence they can be neglected. Therefore, the dynamic stability equation of a beam is written as

$$w,_{xxxx} = \tfrac{1}{D_x}\left(q(x,t) - \rho A\frac{d^2w}{dt^2} + P\frac{d^2w}{dx^2} - M^{TH}_{x,xx}\right) + \tfrac{1}{B_x}N^{TH}_{x,xx} \tag{10}$$

<u>Free Vibration Problem</u>: Free oscillation of the beam from its initial position can be calculated from the equation

$$\frac{d^4w}{dx^4} + \tfrac{\rho A}{D_x}\frac{d^2w}{dt^2} = 0 \tag{11}$$

<u>Buckling problem:</u> Linear buckling loads and modes can be calculated from solving the equation

$$\frac{d^4w}{dx^4} = \tfrac{P}{D_x}\frac{d^2w}{dx^2} \qquad \text{or} \qquad \frac{d^4w}{dx^4} = \tfrac{P}{D_x}\frac{d^2(w+w_o)}{dx^2}$$

Where w_o is the initial deformation caused due to thermal-moisture loading.