4 ANALYSIS OF LAMINATES

4.1 STATIC ANALYSIS

4.1.1 Energy Method of Analysis of Structures

Principle of Total Potential Energy:

If the deformation of an elastic body can be expressed in terms of admissible functions that satisfy the kinematic boundary conditions, then the total potential energy (TPE) of the body is stationary. In other words, the first variation of the TPE with respect to coefficients of the admissible functions is zero.

Total potential energy, $\Pi = U + V = \int_v \int_0^{\varepsilon} \sigma d\varepsilon dv - \int p du$

where U = Strain energy of the elastic body
V = Potential energy due to work done by the applied forces
σ, ε, u, and p are stress, strain, displacement, and the loading, respectively.

Principle: $\dfrac{\partial \Pi}{\partial c_i} = 0$; C_i represent the generalized coordinates .

Procedure:

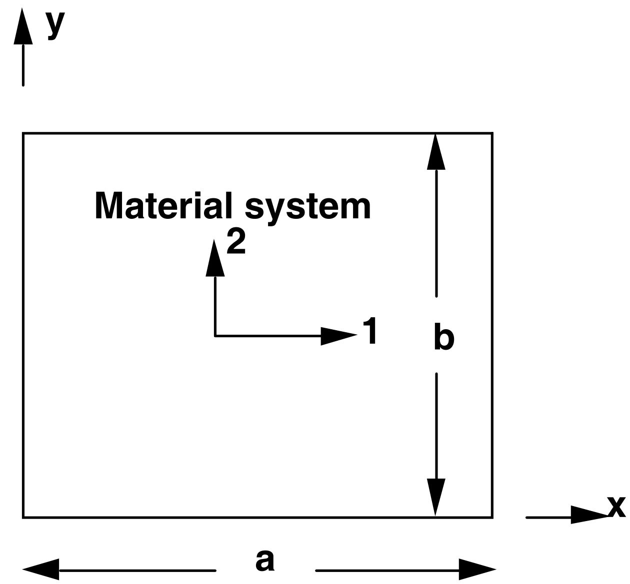
Step 1: Select admissible functions that satisfy the geometric boundary conditions of the problem. Define the displacements as a function of admissible functions and undetermined constants called generalized displacements. $u = c_i \phi_i$.

Step 2: Calculate the total potential energy ($\Pi = U + V$) of the body using the appropriate **strain-displacement and constitutive equations**.

Step 3: Minimize the TPE (Π) with respect to the generalized displacements. This results in a set of algebraic equations to be solved for generalized constants.

Step 4: Number of equations will be equal to number of constants. Then Substitute the constants in the displacement equation. This completes the analysis.

Example: **Clamped-clamped rectangular plate subjected to an uniformly distributed load, -q.**



<u>Boundary Conditions:</u>

$$w = 0;\ \text{and}\ \frac{\partial w}{\partial x} = 0\ @\ x = 0 \text{ and } x = a$$

$$w = 0;\ \text{and}\ \frac{\partial w}{\partial y} = 0\ @\ y = 0 \text{ and } y = b$$

<u>Displacement approximation:</u>

$$w(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \left(1 - \cos \frac{2i\pi x}{a}\right)\left(1 - \cos \frac{2j\pi y}{b}\right)$$

One-term approximation: $w(x,y) = c\left(1 - \cos \frac{2\pi x}{a}\right)\left(1 - \cos \frac{2\pi y}{b}\right)$

Total Potential Energy, Π: (Specially orthotropic symmetric laminate)

$$\Pi(w) = \int_0^b \int_0^a \left\{ \frac{1}{2}\left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 \right] - qw \right\} dx\,dy$$

Substituting for w in the above equation, and performing partial differentiation w.r.to c leads to

$$4ab\pi^4 c\left[\frac{3D_{11}}{a^4} + \frac{2D_{12}}{a^2b^2} + \frac{3D_{22}}{b^4} + \frac{4D_{66}}{a^2b^2}\right] = qab$$

Example:

For a square (a = b) and isotropic plate $\{D_{11} = D_{22} = D, D_{12} = \nu D,$ and $D_{66} = (1-\nu)D/2\}$, the expression for c simplifies to

$$c = \frac{qa^4}{32\pi^4 D}$$

therefore, the expression for w is

$$w(x,y) = \frac{qa^4}{32\pi^4 D}\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi y}{b}\right)$$

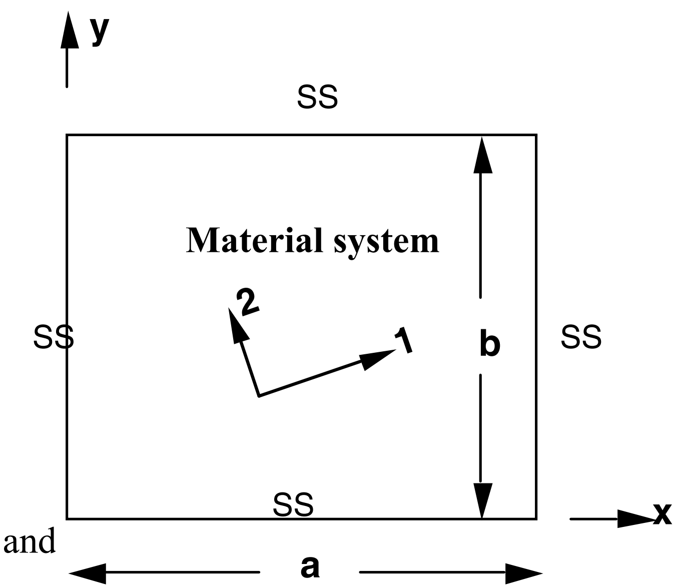
w_{max} is at the center of the plate, $w_{max} = \frac{qa^4}{8\pi^4 D} = 0.00128\frac{qa^4}{D}$

Series solution by Evans (1939):

$w_{max} = 0.00126\frac{qa^4}{D}$; The approximate solution is about 1% larger than the exact solution.

Section 4.2 CLASSICAL METHODS

Analysis of Simply-Supported Laminated Plate Subjected to an Uniform Load (p)



Case 1: Specially Orthotropic Laminate

ŅMaterial axes parallel to the Plate axesÓ - All coupling (Normal-Shear and Bending-Stretching) terms are ZERO.

GDE: $D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} = p$ (1)

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = -D_{11}\, w_{,xx} - D_{12}\, w_{,yy} = 0$

@ y=0 & b: $w = 0$ & $M_y = -D_{12}\, w_{,xx} - D_{22}\, w_{,yy} = 0$

Selection of Displacement Functions: Because the GDE & BCs are even derivatives of x and y, we can select a solution in the form:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (2)$$

Loading:

Different types of loading can be expressed using the Fourier series, as

$$p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (3)$$

The Fourier coefficients are calculated for each type of loading as follows

$$\int_0^a \int_0^b p(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dxdy = \int_0^a \int_0^b \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx$$

$$p_{mn} = \frac{4}{ab} \int_0^a \int_0^b p(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dxdy \qquad (4)$$

For an uniform loading of qo,

$$p_{mn} = \frac{4q_o}{ab} \int_0^a \int_0^b \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dxdy = \frac{16 q_o}{\pi^2 mn}$$, For odd numbers of m & n.

Solution:

Substituting Eqs 2 & 3 in Eq 1 (GDE), we get

$$a_{mn} = \frac{p_{mn}}{\pi^4 \left\{ D_{11}\left(\frac{m}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{a}\right)^4 \right\}}$$

(5)

$$\therefore w = \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \frac{16 q_o \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\pi^6 \left\{ D_{11}\left(\frac{m}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{a}\right)^4 \right\}}$$

(6)

Case 2: Symmetric Angle Ply Laminate

Because of symmetry inplane displacements decouple transverse displacement, however bending-twisting coupling stiffness (D_{16} and D_{26}) are non zero.

GDE: $D_{11}w_{,xxxx} + 4D_{16}w_{,xxxy} + 2(D_{12} + 2D_{66})w_{,xxyy} + 4D_{26}w_{,xyyy} + D_{22}w_{,yyyy} = p(x,y)$ (7)

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = -D_{11}\,w_{,xx} - D_{12}\,w_{,yy} - 2D_{16}\,w_{,xy} = 0$

@ y=0 & b: $w = 0$ & $M_y = -D_{12}\,w_{,xx} - D_{22}\,w_{,yy} - 2D_{26}\,w_{,xy} = 0$

Solution:

Because of non-zero C, it is very difficult to select displacement functions that satisfies both GDE and BCS. Therefore, total potential energy approach is used to solve such problems. In the TPE method, the displacements must satisfy only the kinematic boundary conditions. Minimization of the TPE satisfies the GDE and the force boundary condition integrated over the total boundary.

$$\Pi = \frac{1}{2}\int_0^a\int_0^b \left[D_{11}(w_{,xx})^2 + 4D_{16}w_{,xx}w_{,xy} + 2D_{12}w_{,xx}w_{,yy} + 2D_{66}(w_{,xy})^2 + 4D_{26}w_{,yy}w_{,xy} + D_{22}(w_{,yy})^2 - 2pw\right]dxdy$$

Assume

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Substituting w function and minimizing Π w.r.t a_{mn} leads to m*n set of equations. Solution of these equations will give the coefficients a_{mn}.

For $D_{22}/D_{11} = 1$, $(D_{12} + 2D_{66})/D_{11} = 1.5$ and

$D_{16}/D_{11} = D_{26}/D_{11} = -0.5$; $m = n = 7$

$$w_{\max} = \frac{0.00425 a^4 p}{D_{11}};$$

Equivalent Orthotropic sol. (D_{16} and $D_{26} = 0$)

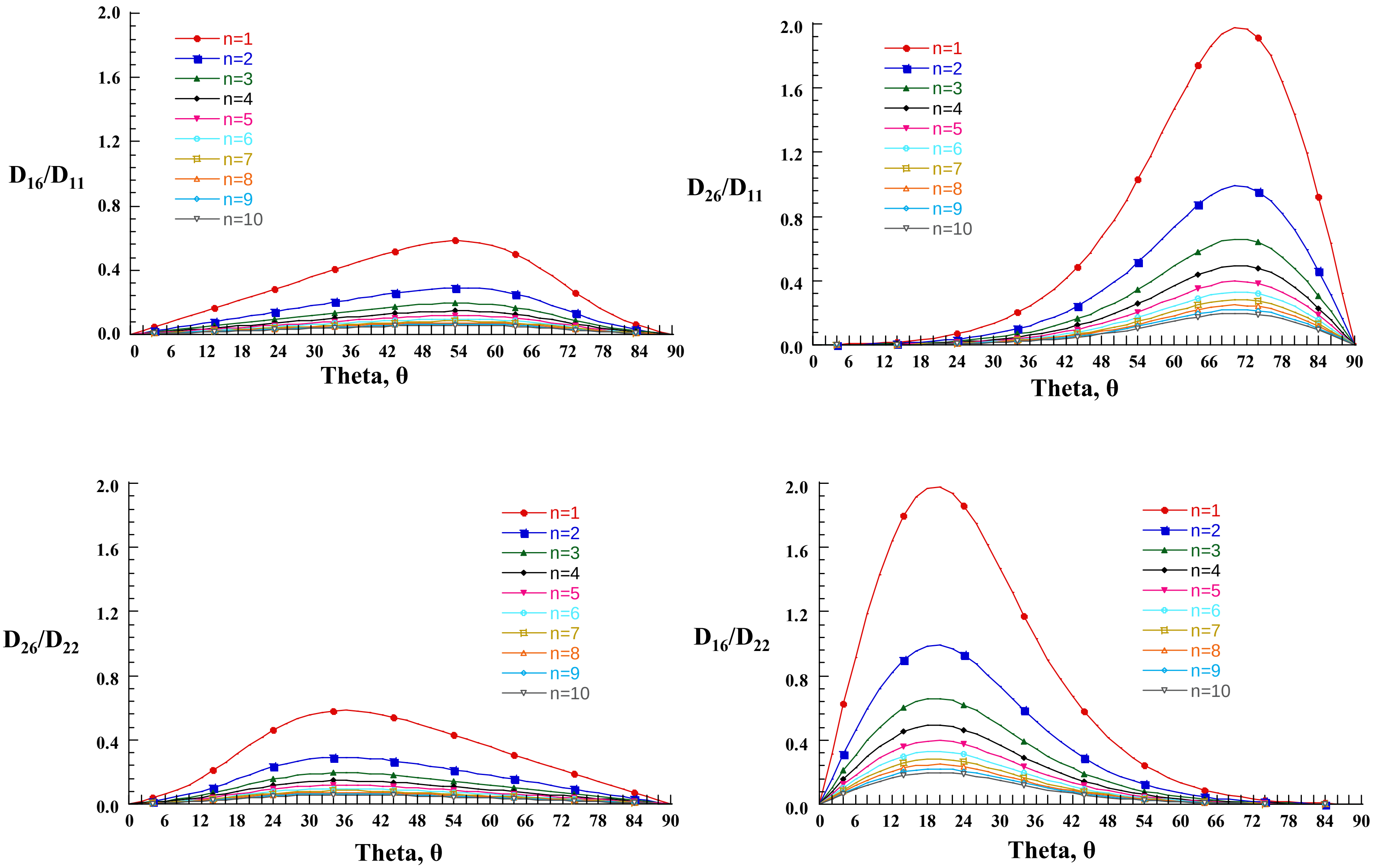
$$(w_{\max})_{EqiOrtho} = \frac{0.00324 a^4 p}{D_{11}}$$

Exact solution: $$(w_{\max})_{Exact} = \frac{0.00452 a^4 p}{D_{11}}$$

* J. E. Ashton, "An Analogy for Certain anisotropic Plates.", *J. Composite Materials*, Vol. 3, 1969, pp. 355-358.

Symmetric Angle-ply Laminate (AS4 Carbon/epoxy)

(θ/-θ)ns AS4/3501-6 Carbon/Epoxy Laminate



Case 3: Anti-Symmetric Cross- Ply Laminate

Properties:

$A_{11} = A_{22}$ and $D_{11} = D_{22}$, and $B_{22} = -B_{11}$

Other non zero terms are: A_{12}, A_{66}, D_{12}, and D_{66}.

Normal-stretching and bending-twisting terms are zero.

GDE:

$$A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} - B_{11}w_{,xxx} = 0 \tag{8}$$

$$(A_{12} + A_{66})u_{,xy} + A_{66}v_{,xx} + A_{11}v_{,yy} + B_{11}w_{,yyy} = 0 \tag{9}$$

$$D_{11}(w_{,xxxx} + w_{,yyyy}) + 2(D_{12} + 2D_{66})w_{,xxyy} - B_{11}(u_{,xxx} - v_{,yyy}) = p \tag{10}$$

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = B_{11}u_{,x} - D_{11} w_{,xx} - D_{12} w_{,yy} = 0$

$v = 0$ & $N_x = A_{11}u_{,x} + A_{12}v_{,y} - B_{11}w_{,xx} = 0$

@ y=0 & b: $w = 0$ & $M_y = -B_{11}v_{,y} - D_{12} w_{,xx} - D_{22} w_{,yy} = 0$

$u = 0$ & $N_y = A_{12}u_{,x} + A_{22}v_{,y} + B_{11}w_{,yy} = 0$

Displacement Functions:

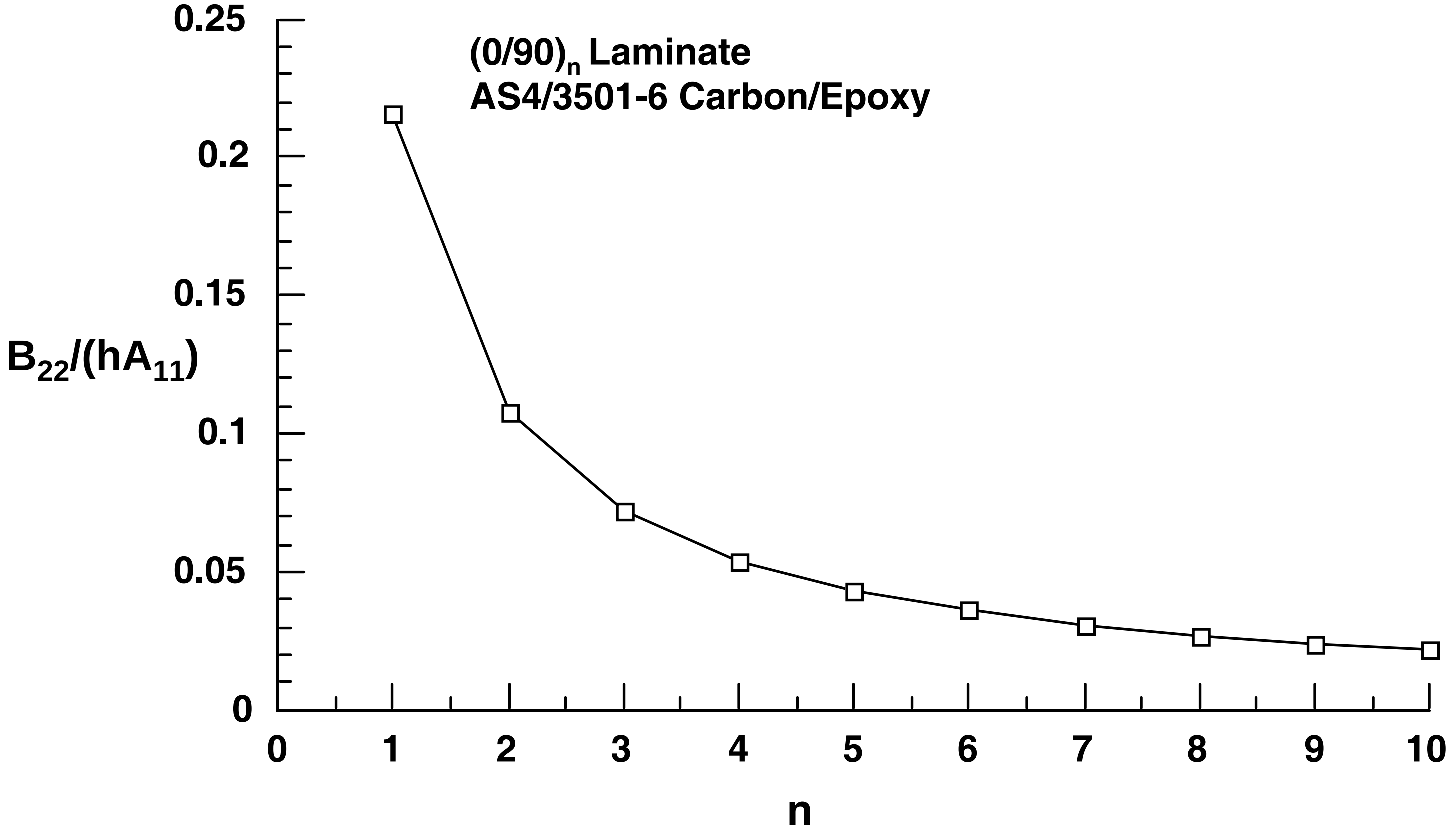
$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$v = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Loading: $p = q_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$

Anti-Symmetric Cross-Ply Laminate



Case 4: Anti-Symmetric Angle-Ply Laminate

Properties:

Non Zero Extensional Stiffness: A_{11}, A_{22}, A_{12} and $A_{66.}$

Non Zero Flexural Stiffness: $D_{11,}$ D_{22}, D_{12}, and D_{66}.

Bending-Extensional Coupling Stiffness: B_{16} and B_{26}

GDE:

$$A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} - 3B_{16}w_{,xxy} - B_{26}w_{,yyy} = 0 \quad (11)$$

$$(A_{12} + A_{66})u_{,xy} + A_{66}v_{,xx} + A_{22}v_{,yy} - B_{16}w_{,xxx} - 3B_{26}w_{,xyy} = 0 \quad (12)$$

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} - B_{16}(3u_{,xxy} + v_{,xxx}) - B_{26}(u_{,yyy} + 3v_{,xyy}) = p \quad (13)$$

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = B_{16}(u_{,y} + v_{,x}) - D_{11}w_{,xx} - D_{12}w_{,yy} = 0$

$u = 0$ & $N_{xy} = A_{66}(u_{,y} + v_{,x}) - B_{16}w_{,xx} - B_{26}w_{,yy} = 0$

@ y=0 & b: $w = 0$ & $M_y = B_{26}(u_{,y} + v_{,x}) - D_{12}w_{,xx} - D_{22}w_{,yy} = 0$

$v = 0$ & $N_{xy} = A_{66}(u_{,y} + v_{,x}) - B_{16}w_{,xx} - B_{26}w_{,yy} = 0$

Displacement Function

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Loading:

$$\boldsymbol{p = p_o Sin\frac{\pi x}{a} Sin\frac{\pi y}{b}}$$

$$T_{11} = A_{11}(\frac{m\pi}{a})^2 + A_{66}(\frac{n\pi}{b})^2$$

$$T_{12} = (A_{12} + A_{66})(\frac{m\pi}{a})(\frac{n\pi}{b})$$

$$T_{13} = -[3B_{16}(\frac{m\pi}{a})^2 + B_{26}(\frac{n\pi}{b})^2](\frac{n\pi}{b})$$

$$T_{22} = A_{66}(\frac{m\pi}{a})^2 + A_{22}(\frac{n\pi}{b})^2$$

$$T_{23} = -[B_{16}(\frac{m\pi}{a})^2 + 3B_{26}(\frac{n\pi}{b})^2](\frac{m\pi}{a})$$

$$T_{33} = D_{11}(\frac{m\pi}{a})^4 + 2(D_{12} + 2D_{66})(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + D_{22}(\frac{n\pi}{b})^4$$

$$A_{mn} = \frac{T_{12}T_{23} - T_{22}T_{13}}{\Delta} p_0$$

$$B_{mn} = \frac{T_{12}T_{13} - T_{11}T_{23}}{\Delta} p_0$$

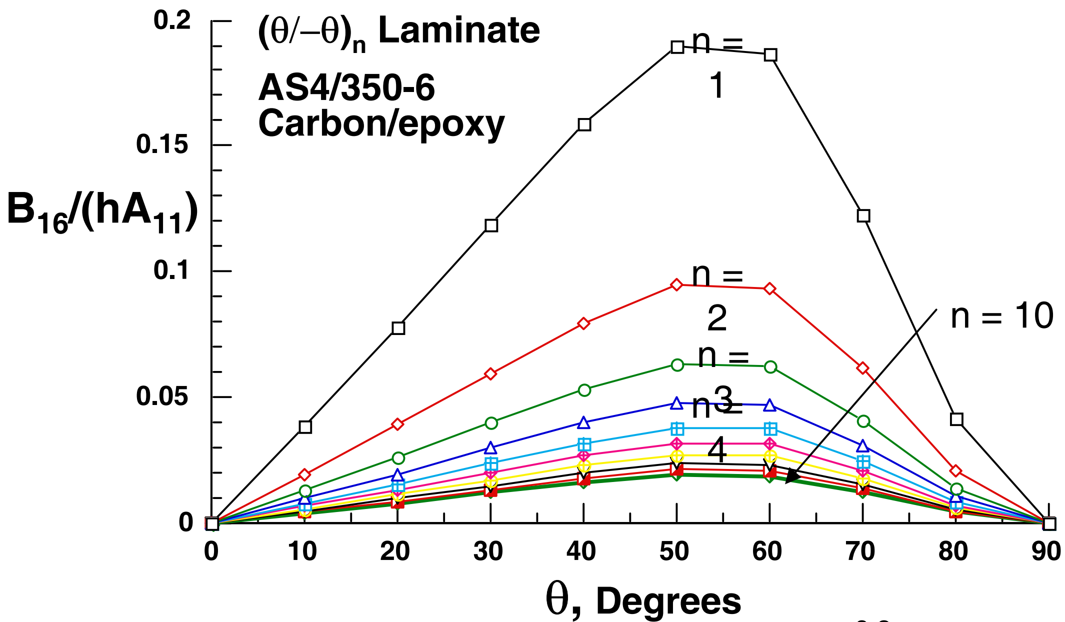
$$C_{mn} = \frac{T_{11}T_{22} - T_{12}{}^2}{\Delta} p_0$$

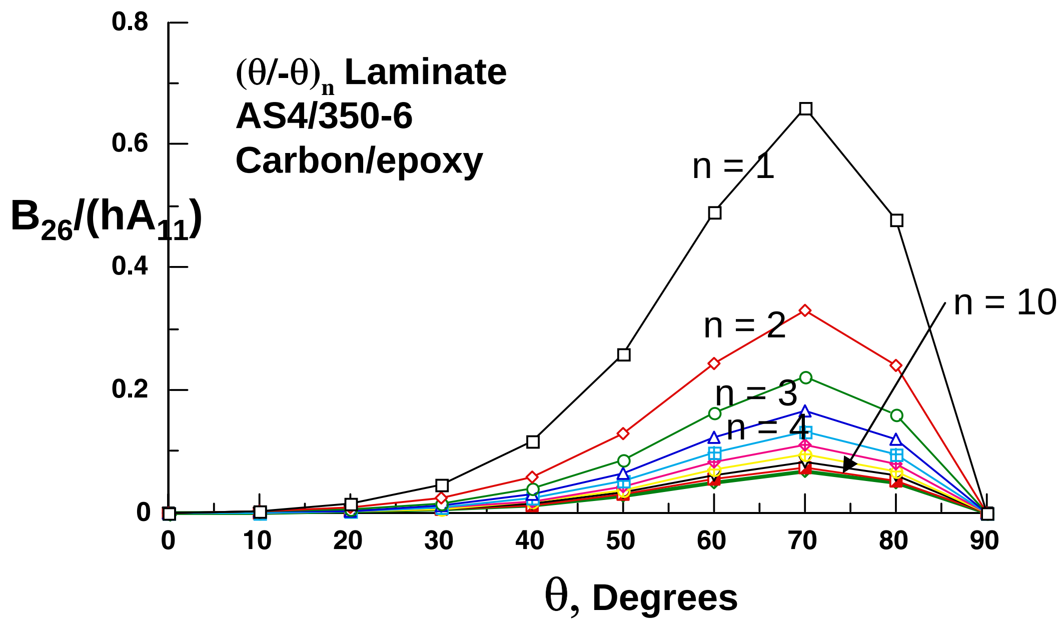
where

$$\Delta = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{vmatrix}$$

Anti-Symmetric Angle-Ply Laminate

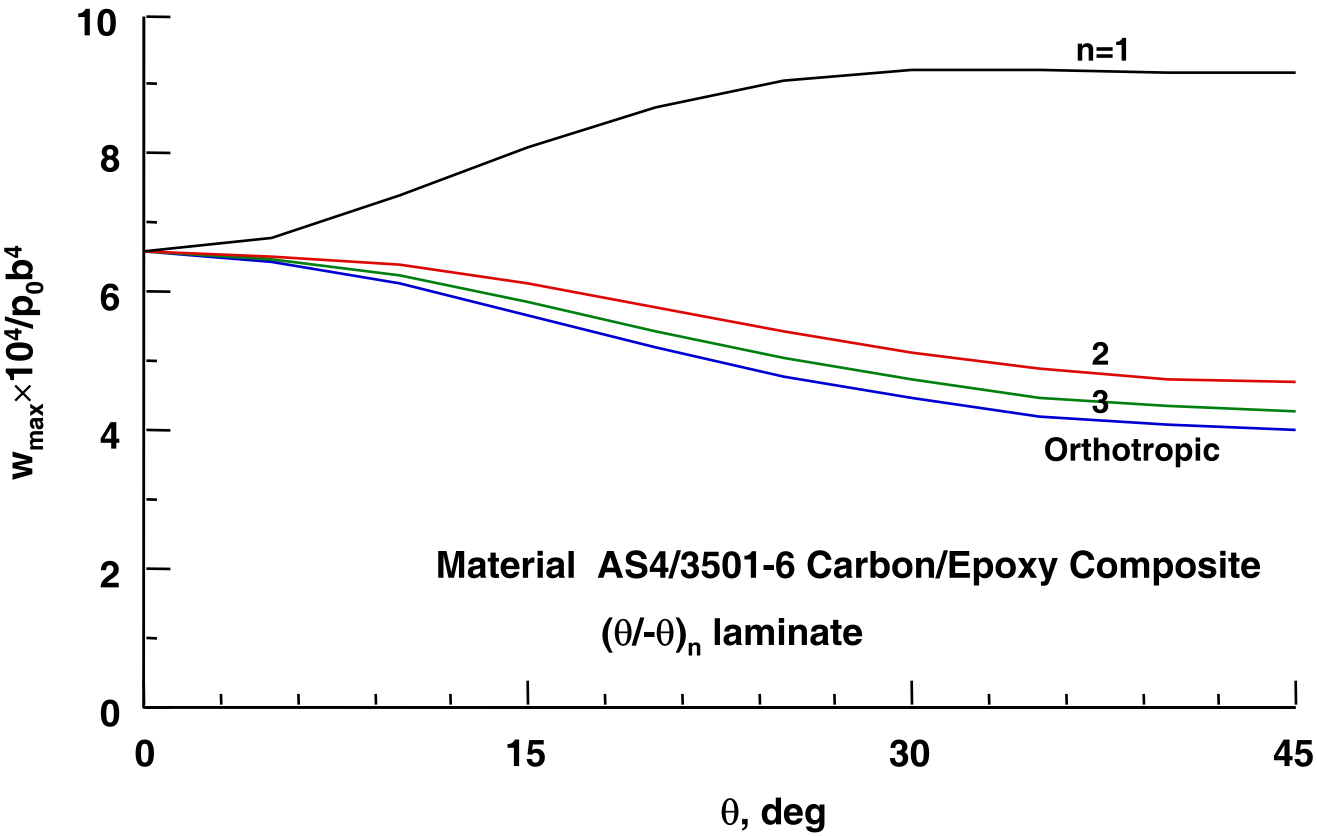
$(\theta/-\theta)_n$ AS4/3501-6 Laminate





VARIATION OF W_{MAX} WITH θ

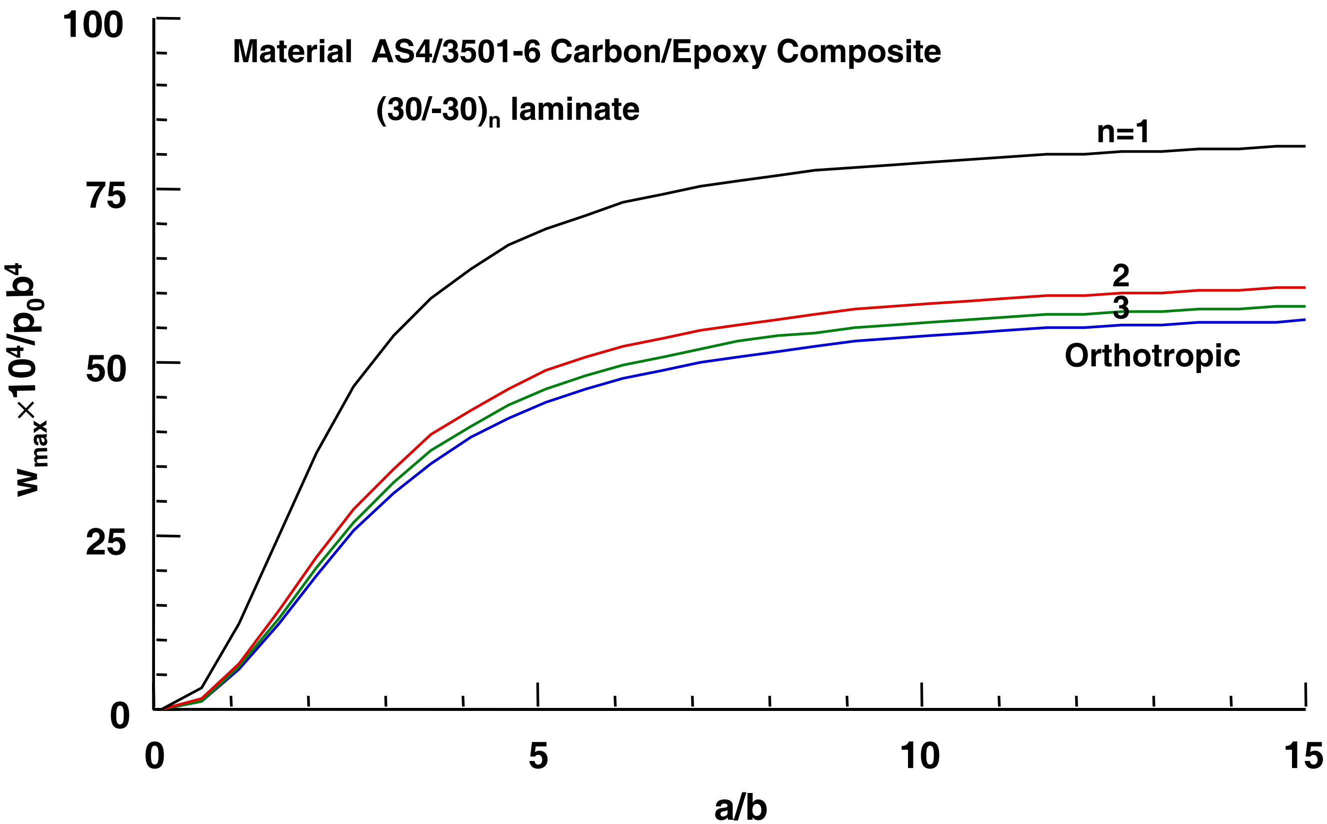
Anti-Symmetric Angle-Ply laminate (a/b=1)



Anil Bhragava

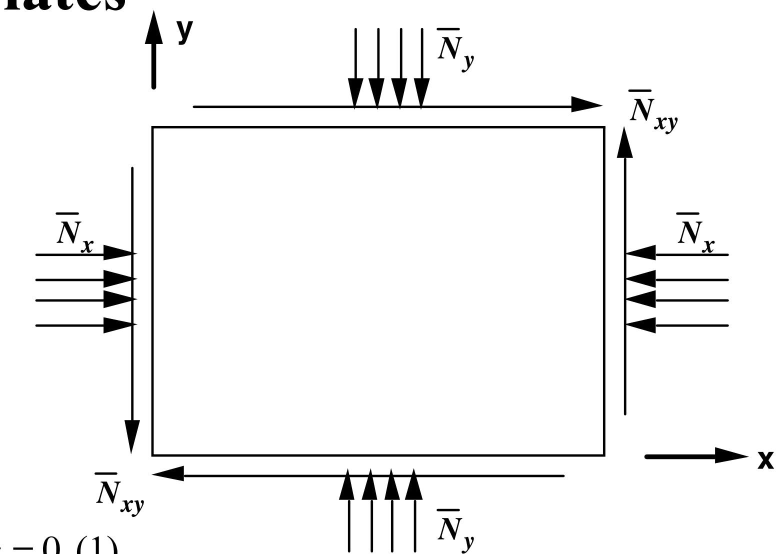
Variation of W_{MAX} With a/b Ratio

Anti-Symmetric Angle-Ply laminate



4.3 Buckling Analysis of Laminated Plates

4.3.1 Buckling Equations of Equilibrium.



$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0$$

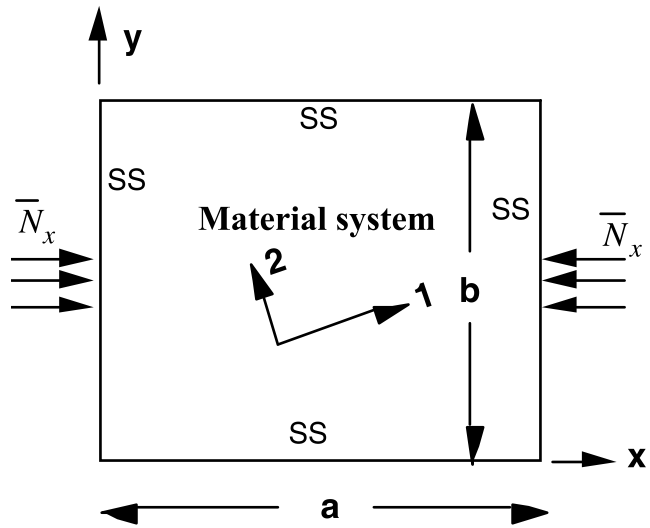
$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \bar{N}_x \frac{\partial^2 w}{\partial x^2} + \bar{N}_y \frac{\partial^2 w}{\partial y^2} + 2\bar{N}_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (1)$$

Where $\bar{N}_x$, $\bar{N}_y$, *and* $\bar{N}_{xy}$ are the edge loads.

If the prebuckling state of the laminate is not flat we can re-write the above equation in a variation of the prebuckled state.

4.3.2 Buckling of S-S laminated Plate Subjected to $\overline{N}_x$ Loading.



Case (1): Symmetric Specially Orthotropic laminate

GDE: $D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} + \overline{N}_x w_{,xx} = 0$ (2)

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = -D_{11}\, w_{,xx} - D_{12}\, w_{,yy} = 0$

@ y=0 & b: $w = 0$ & $M_y = -D_{12}\, w_{,xx} - D_{22}\, w_{,yy} = 0$

Selection of Displacement Functions: Because the GDE & Bcs are even derivatives of x and y, we can select a solution in the form:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (3)$$

Where m and n are number of buckled half-waves in x- and y-directions respectively. Substituting for ŌwÕ in the GDE, we get for a non-trivial solution

$$\left(\overline{N}_x\right)_{mn} = \pi^2 \left\{ D_{11}\left(\frac{m}{a}\right)^2 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4\left(\frac{a}{m}\right)^2 \right\} \quad (4)$$

Buckling load is a function of
(i) Elastic properties of the material.
(ii) Geometry (b and aspect ratio, a/b)
(iii) Number of half-waves in x- and y-directions.

Minimum buckling load occurs @ n = 1.

$$\left(\overline{N}_x\right)_{m} = \frac{\pi^2}{b^2} \left\{ D_{11}\left(\frac{mb}{a}\right)^2 + 2\left(D_{12} + 2D_{66}\right) + D_{22}\left(\frac{a}{bm}\right)^2 \right\} \quad (5)$$

Minimization of the above Eq 5 w.r.to aspect ratio (a/b), we get

$$\left(\overline{N}_x\right)_{\min} = \frac{2\pi^2}{b^2} \left\{ \sqrt{D_{11}D_{22}} + \left(D_{12} + 2D_{66}\right) \right\} \quad (6)$$

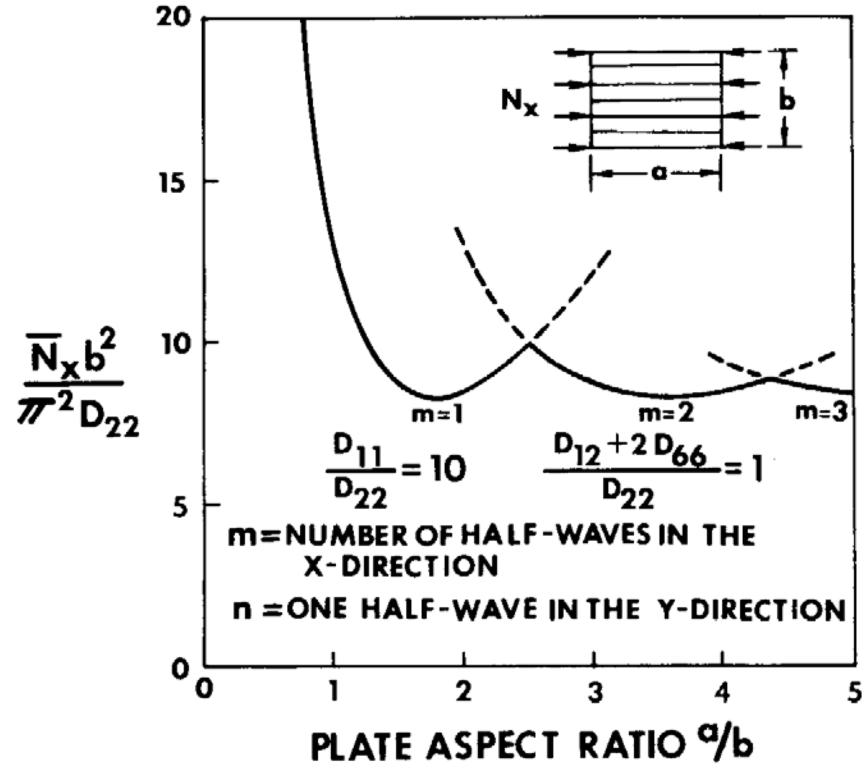
for $a/b = m\sqrt[4]{D_{11}/D_{22}}$. Notice that Eq 6 is independent of length (a) and m.

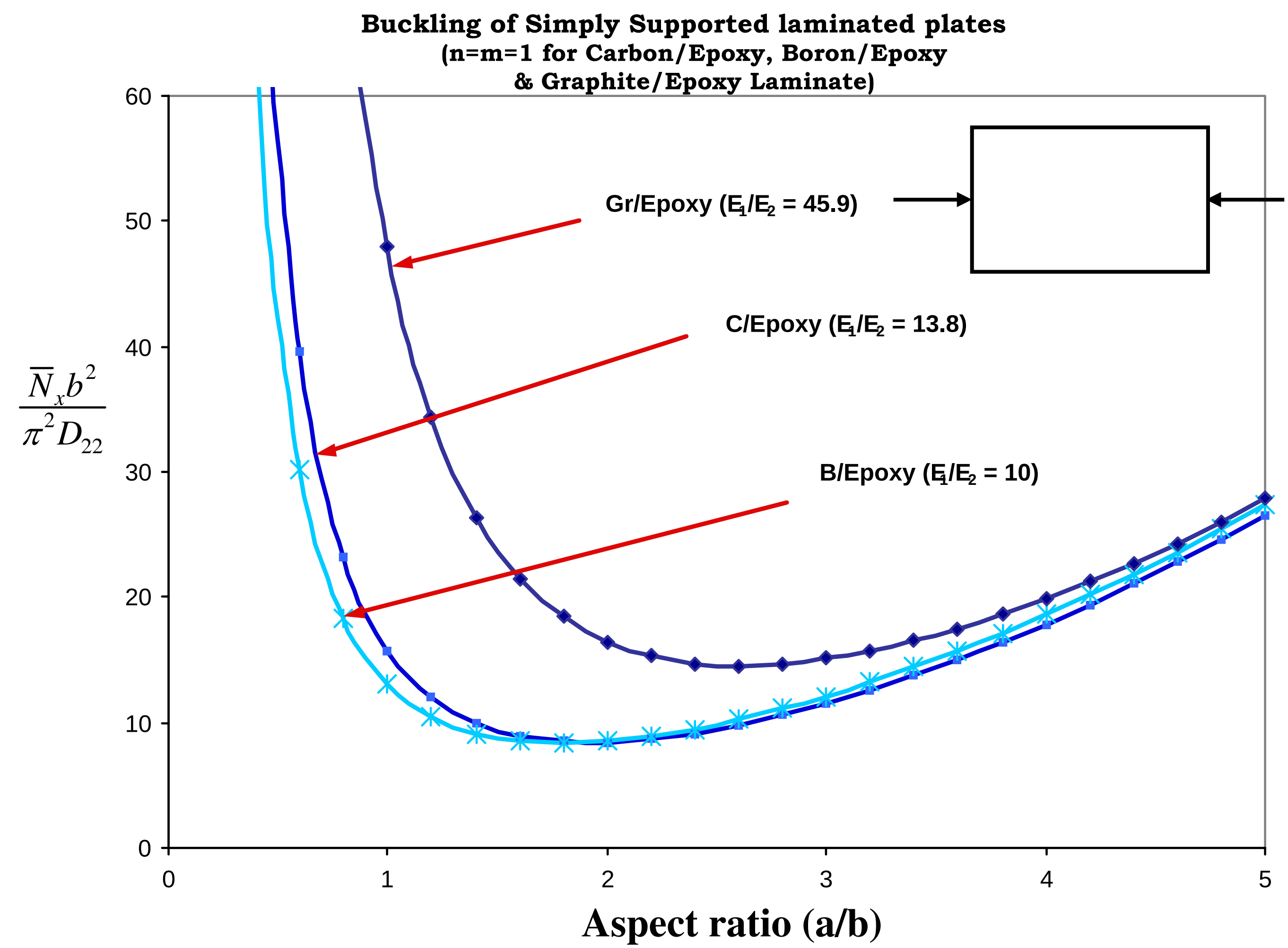
Example: For $D_{11}/D_{22} = 10$ and $(D_{12} + 2D66)/D_{22} = 1$

$$\left(\overline{N}_x\right)_{\min} = \frac{2\pi^2 D_{22}}{b^2}\left\{\sqrt{10} + 1\right\}$$

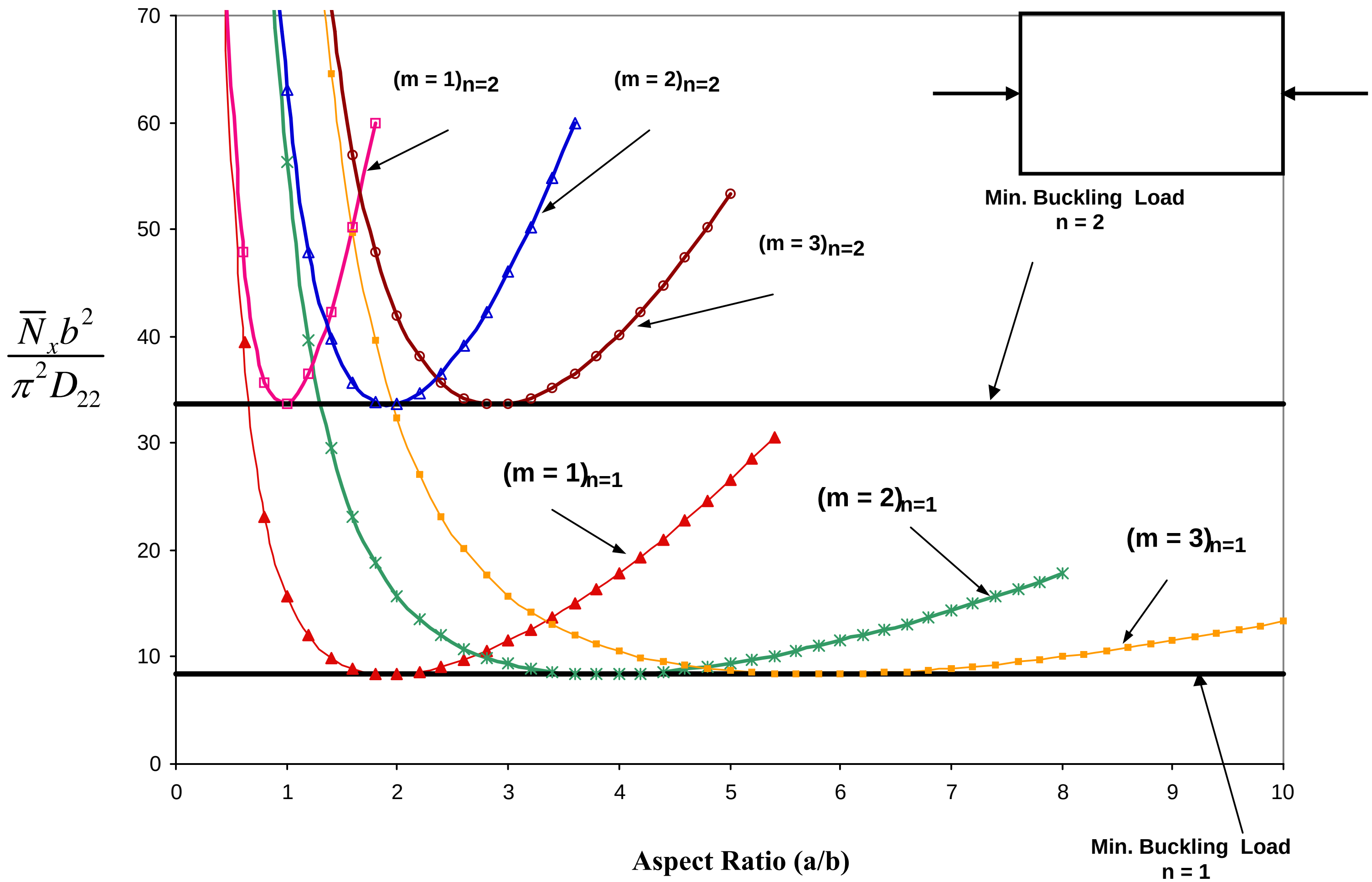
Isotropic Plate:

$$\left(\overline{N}_x\right)_{Iso} = \frac{\pi^2 D}{b^2}\left\{\left(\frac{mb}{a}\right)^2 + 2 + \left(\frac{a}{bm}\right)^2\right\}$$




Buckling of Simply Supported laminated plates
(n=m=1 for Carbon/Epoxy, Boron/Epoxy
& Graphite/Epoxy Laminate)
Gr/Epoxy (E_1/E_2 = 45.9)
C/Epoxy (E_1/E_2 = 13.8)
B/Epoxy (E_1/E_2 = 10)
$\frac{\overline{N}_x b^2}{\pi^2 D_{22}}$
Aspect ratio (a/b)
0
10
20
30
40
50
60
0
1
2
3
4
5

Buckling of Simply Supported laminated plates under in-plane loa
Carbon/Epoxy Laminate



Case 2: Symmetric Angle Ply Laminate

Because of symmetry inplane displacements decouple transverse displacement, however bending-twisting coupling stiffness (D_{16} and D_{26}) are non zero.

GDE: $D_{11}w_{,xxxx} + 4D_{16}w_{,xxxy} + 2(D_{12} + 2D_{66})w_{,xxyy} + 4D_{26}w_{,xyyy} + D_{22}w_{,yyyy} + \overline{N}_x w_{,xx} = 0$ (7)

Boundary Conditions:

@ x=0 & a: w = 0 & $M_x = -D_{11}\,w_{,xx} - D_{12}\,w_{,yy} - 2D_{16}\,w_{,xy} = 0$

@ y=0 & b: w = 0 & $M_y = -D_{12}\,w_{,xx} - D_{22}\,w_{,yy} - 2D_{26}\,w_{,xy} = 0$

Solution: Energy method is used to obtain the solution.

Assumed displacements:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Case 3: Anti-Symmetric Cross- Ply Laminate

Properties:

$A_{11} = A_{22}$ and $D_{11} = D_{22}$, and $B_{22} = -B_{11}$

Other non zero terms are: A_{12}, A_{66}, D_{12}, and D_{66}.

Normal-stretching and bending-twisting terms are zero.

GDE:

$$A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} - B_{11}w_{,xxx} = 0 \qquad (8)$$

$$(A_{12} + A_{66})u_{,xy} + A_{66}v_{,xx} + A_{11}v_{,yy} + B_{11}w_{,yyy} = 0 \qquad (9)$$

$$D_{11}(w_{,xxxx} + w_{,yyyy}) + 2(D_{12} + 2D_{66})w_{,xxyy} - B_{11}(u_{,xxx} - u_{,yyy}) + \overline{N}_x w_{,xx} = 0 \qquad (10)$$

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = B_{11}u_{,x} - D_{11}w_{,xx} - D_{12}w_{,yy} = 0$

$v = 0$ & $N_x = A_{11}u_{,x} + A_{12}v_{,y} - B_{11}w_{,xx} = 0$

@ y=0 & b: $w = 0$ & $M_y = -B_{11}v_{,y} - D_{12}w_{,xx} - D_{22}w_{,yy} = 0$

$u = 0$ & $N_y = A_{12}u_{,x} + A_{22}v_{,y} + B_{11}w_{,yy} = 0$

Displacement Functions:

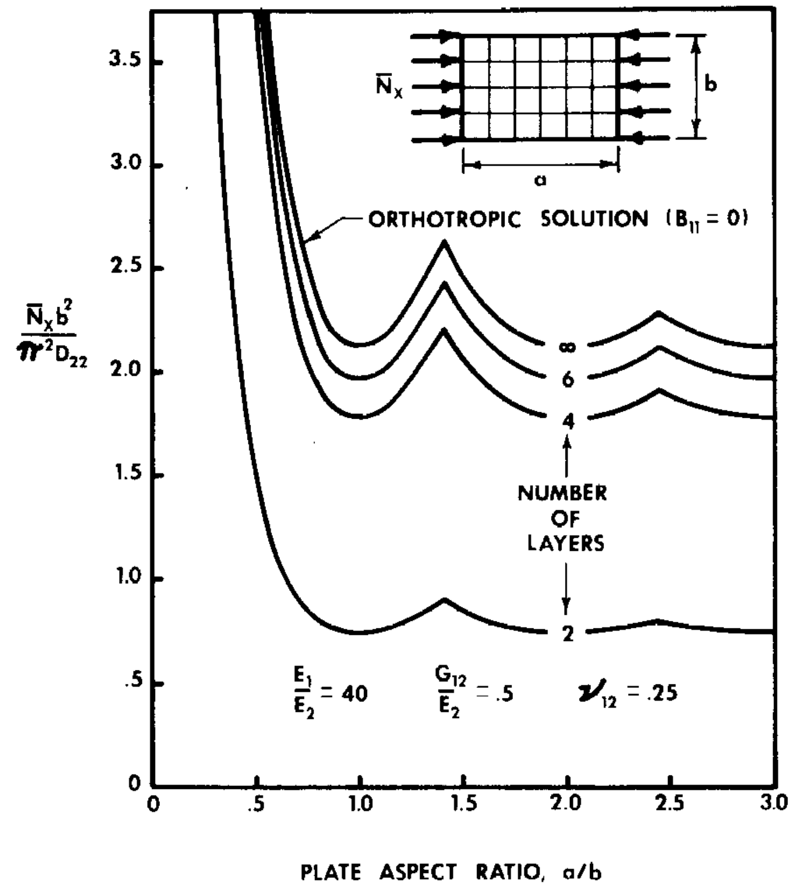
$$u = \bar{u}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$v = \bar{v}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad (11)$$
$$w = \bar{w}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

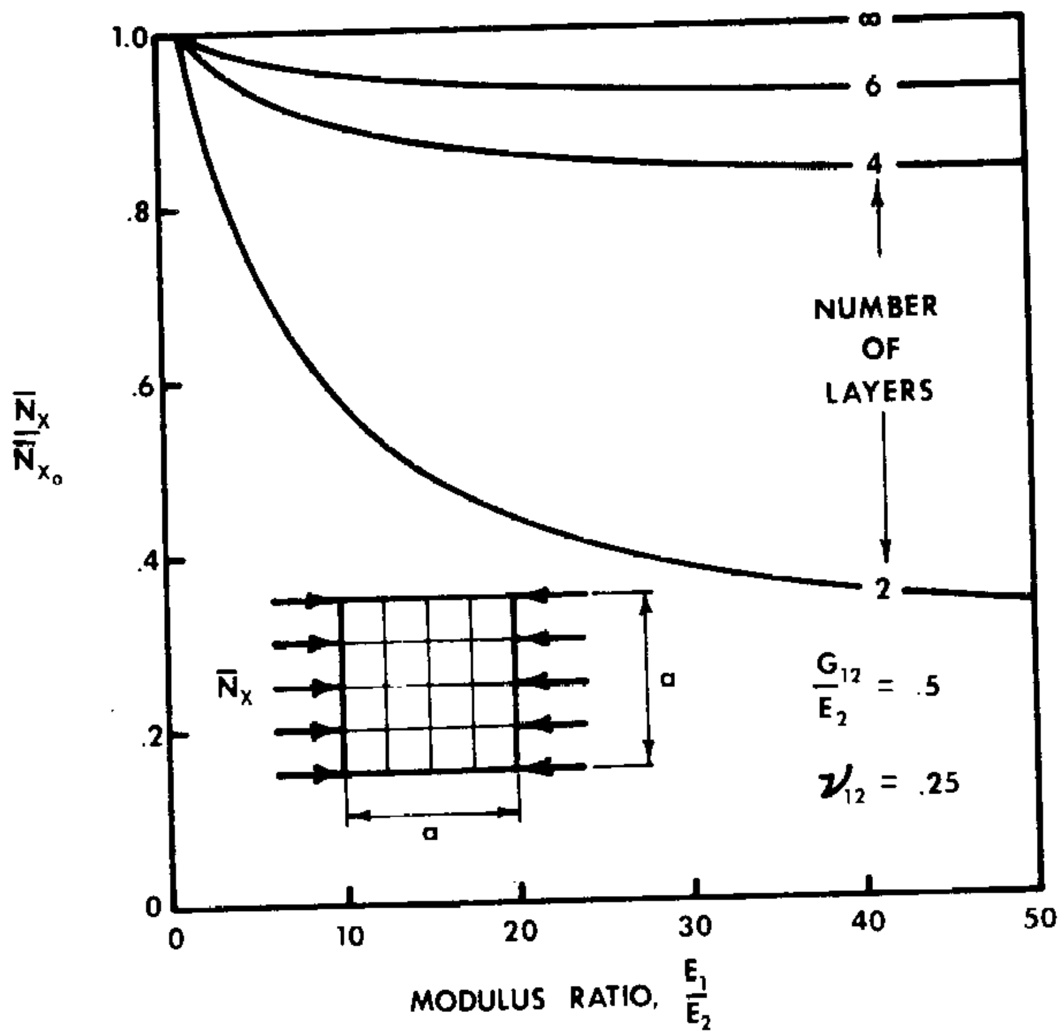
Buckling solution:

$$\left(\bar{N}_x\right)_{mn} = \left(\frac{a}{m\pi}\right)^2 \left\{T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right\} \qquad (12)$$

Where

$$T_{11} = A_{11}\left(\frac{m\pi}{a}\right)^2 + A_{66}\left(\frac{n\pi}{b}\right)^2$$
$$T_{12} = \left(A_{12} + A_{66}\right)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)$$
$$T_{13} = -B_{11}\left(\frac{m\pi}{a}\right)^3$$
$$T_{22} = A_{11}\left(\frac{n\pi}{b}\right)^2 + A_{66}\left(\frac{m\pi}{a}\right)^2 \qquad (13)$$
$$T_{23} = B_{11}\left(\frac{n\pi}{b}\right)^3$$
$$T_{33} = D_{11}\left\{\left(\frac{m\pi}{a}\right)^4 + \left(\frac{n\pi}{b}\right)^4\right\} + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2$$





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$\frac{G_{12}}{E_2}$ = .5
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MODULUS RATIO, $\frac{E_1}{E_2}$

Case 4: Anti-Symmetric Angle-Ply Laminate

Properties:

Non Zero Extensional Stiffness: A_{11}, A_{22}, A_{12} and $A_{66.}$

Non Zero Flexural Stiffness: $D_{11,}$ D_{22}, D_{12}, and D_{66}.

Bending-Extensional Coupling Stiffness: B_{16} and B_{26}

GDE:

$$A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} - 3B_{16}w_{,xxy} - B_{26}w_{,yyy} = 0 \qquad (14)$$

$$(A_{12} + A_{66})u_{,xy} + A_{66}v_{,xx} + A_{22}v_{,yy} - B_{16}w_{,xxx} - 3B_{26}w_{,xyy} = 0 \qquad (15)$$

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} - B_{16}(3u_{,xxy} + v_{,xxx}) - B_{26}(u_{,yyy} + 3v_{,xyy}) + \overline{N}_x w_{,xx} = 0 \qquad (16)$$

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = B_{16}(u_{,y} + v_{,x}) - D_{11}w_{,xx} - D_{12}w_{,yy} = 0$

$u = 0$ & $N_{xy} = A_{66}(u_{,y} + v_{,x}) - B_{16}w_{,xx} - B_{26}w_{,yy} = 0$

@ y=0 & b: $w = 0$ & $M_y = B_{26}(u_{,y} + v_{,x}) - D_{12}w_{,xx} - D_{22}w_{,yy} = 0$

$v = 0$ & $N_{xy} = A_{66}(u_{,y} + v_{,x}) - B_{16}w_{,xx} - B_{26}w_{,yy} = 0$

Displacement Functions:

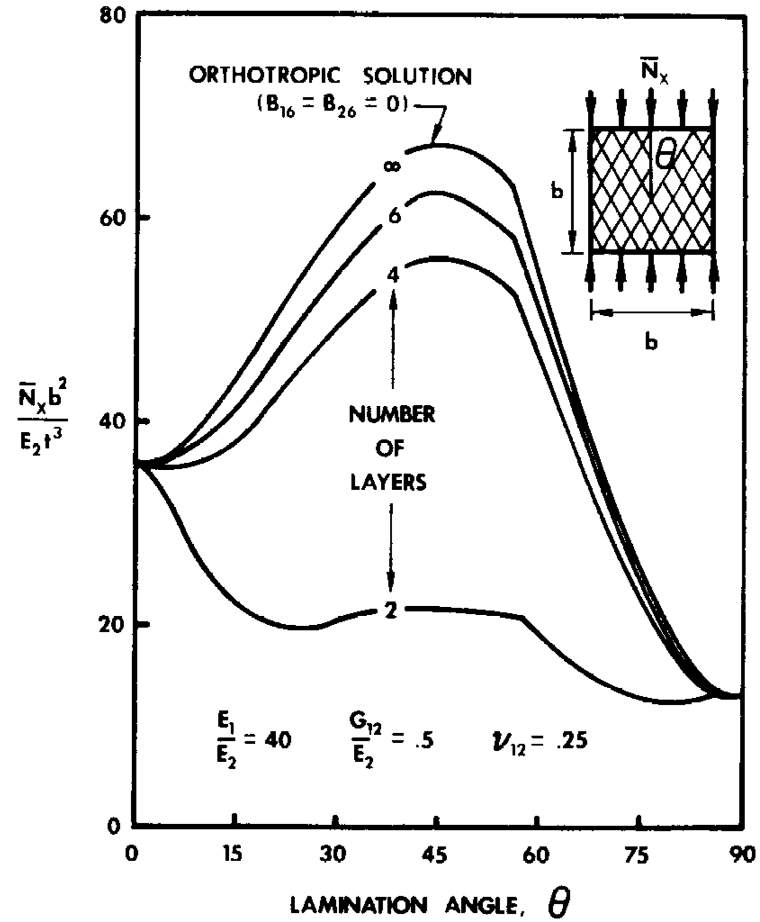
$$u = \bar{u}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$
$$v = \bar{v}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (17)$$
$$w = \bar{w}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

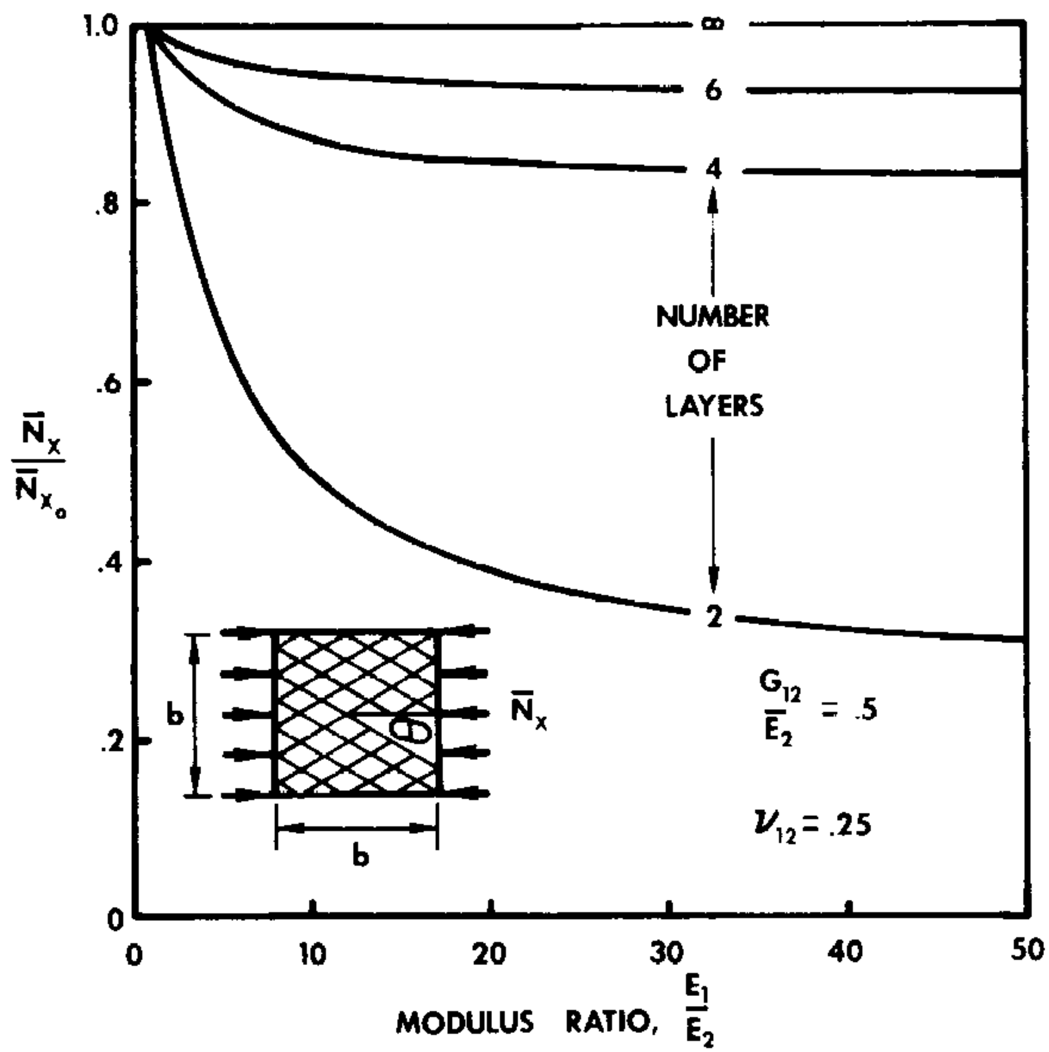
Buckling Solution:

$$\left(\bar{N}_x\right)_{mn} = \left(\frac{a}{m\pi}\right)^2\left\{T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right\} \qquad (18)$$

Where

$$T_{11} = A_{11}\left(\frac{m\pi}{a}\right)^2 + A_{66}\left(\frac{n\pi}{b}\right)^2$$
$$T_{12} = \left(A_{12} + A_{66}\right)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)$$
$$T_{13} = -\left\{3B_{16}\left(\frac{m\pi}{a}\right)^2 + B_{26}\left(\frac{n\pi}{b}\right)^2\right\}\left(\frac{n\pi}{b}\right)$$
$$T_{22} = A_{22}\left(\frac{n\pi}{b}\right)^2 + A_{66}\left(\frac{m\pi}{a}\right)^2 \qquad (19)$$
$$T_{23} = -\left\{B_{16}\left(\frac{m\pi}{a}\right)^2 + 3B_{26}\left(\frac{n\pi}{b}\right)^2\right\}\left(\frac{m\pi}{a}\right)$$
$$T_{33} = D_{11}\left(\frac{m\pi}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4$$





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$\frac{G_{12}}{E_2} = .5$
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$\bar{N}_x$
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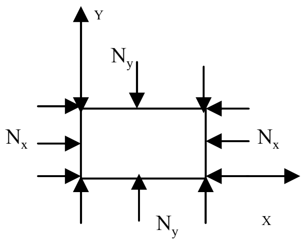
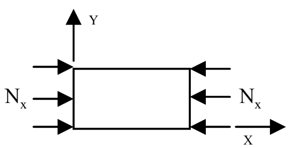
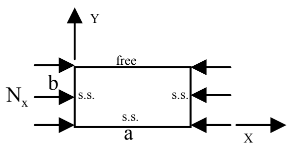
MODULUS RATIO, $\frac{E_1}{E_2}$

Loads	Equation No.	Edge Conditions	Buckling Loads	Comments (Assumptions)
Biaxial Compression 	2.2.2-14	**All sides S-S**	$$N_{x,cr} = \frac{\pi^2}{b^2}\left\{\frac{D_{11}m^4\left(\frac{b}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)m^2n^2\left(\frac{b}{a}\right)^2 + D_{22}n^4}{m^2\left(\frac{b}{a}\right)^2 + \Phi n^2}\right\}_{,\min}$$	1. $\Phi = \frac{N_y}{N_x}$ $m, n = 1, 2, ..., \infty$ 2.Equation to be minimized with respect to 3.a/b finite 4. Reference 2.2-20
Uniaxial Compression 	2.2.2-15	**All sides S-S**	$$N_{x,cr} = \frac{\pi^2}{b^2}\left\{D_{11}m^2\left(\frac{b}{a}\right)^2 + 2\left(D_{12} + 2D_{66}\right)n^2 + D_{22}\frac{n^4}{m^2}\left(\frac{b}{a}\right)^2\right\}_{,\min}$$	$m, n = 1, 2, ..., \infty$ 1. Equation to be minimized with respect to 2.a/b finite 3. Reference 2.2-20
	2.2.2-16	**All sides S-S**	$$N_{x,cr} = 2\frac{\pi^2}{b^2}\left\{\sqrt{D_{11}D_{22}} + D_{12} + 2D_{66}\right\}$$	1.a/b=infinite 2. Reference 2.2-21
	2.2.2-17	**All sides fixed**	$$N_{x,cr} = \frac{\pi^2}{b^2}\left\{D_{11}m^2\left(\frac{b}{a}\right)^2 + 2.67D_{12} + 5.33\left[D_{22}\frac{1}{m^2}\left(\frac{a}{b}\right)^2 + D_{66}\right]\right\}$$	1.a/b=finite 2.Minimized 3. Reference 2.2-21
	2.2.2-18	**All sides fixed**	$$N_{x,cr} = \frac{\pi^2}{b^2}\left\{4.6\sqrt{D_{11}D_{22}} + 2.67D_{12} + 5.33D_{66}\right\}$$	1.a/b=infinite 2. Reference 2.2-21

Uniaxial compression 	2.2.2-19	**Three sides S-S and one side free**	$N_{x,cr} = K_s \frac{D_{11}}{b^2}$ Where Ks is found from the solution of the transcendental equation $\bar{\beta}\left[\bar{\beta}^2 + \left(\frac{m\pi b}{a}\right)^2 \gamma\right]\left[\bar{\alpha}^2 - \left(\frac{m\pi b}{a}\right)^2 \nu_{xy}\right]\tan\bar{\alpha} =$ $\bar{\alpha}\left[\bar{\alpha}^2 - \left(\frac{m\pi b}{a}\right)^2 \gamma\right]\left[\bar{\beta}^2 - \left(\frac{m\pi b}{a}\right)^2 \nu_{xy}\right]\tan\bar{\beta}$ With $\bar{\beta} = \sqrt{\frac{m\pi b}{a}}\left\{\left[\left(\frac{D_3}{D_{22}}\right)^2\left(\frac{m\pi b}{a}\right)^2 - \left(\frac{D_{11}}{D_{22}}\right)^2\left(\frac{m\pi b}{a}\right)^2 + \left(\frac{N_x b^2}{D_{11}}\right)\left(\frac{D_{11}}{D_{22}}\right)\right]^{\frac{1}{2}} - \left(\frac{D_3}{D_{22}}\right)\left(\frac{m\pi b}{a}\right)\right\}^{\frac{1}{2}}$ $\bar{\alpha} = \sqrt{\frac{m\pi b}{a}}\left\{\left[\left(\frac{D_3}{D_{22}}\right)^2\left(\frac{m\pi b}{a}\right)^2 - \left(\frac{D_{11}}{D_{22}}\right)^2\left(\frac{m\pi b}{a}\right)^2 + \left(\frac{N_x b^2}{D_{11}}\right)\left(\frac{D_{11}}{D_{22}}\right)\right]^{\frac{1}{2}} + \left(\frac{D_3}{D_{22}}\right)\left(\frac{m\pi b}{a}\right)\right\}^{\frac{1}{2}}$ $D_3 = D_{12} + 2D_{66}, \gamma = \frac{4G_{xy}\left(1-\nu_{xy}\nu_{yx}\right)}{E_y} + \nu_{xy}$	1.a/b=finite 2.Minimize Ks with respect to $m, n = 1, 2, \ldots, \infty$ 3. Reference 2.2-19
	2.2.2-20	**Three sides S-S one side free**	$N_{x,cr} = h^3\left[\frac{G_{xy}}{b^2} + \frac{m^2\pi^2 E_x}{12a^2(1-\nu_{xy}\nu_{yx})}\right]$ $N_{x,cr} = \frac{h^3 G_{xy}}{b^2}, \qquad \frac{a}{b} = \infty$	M=1 is minimum Reference 2.2-36

4.4 Free Vibration Analysis of Laminated Plates

4.4.1 Equations of Equilibrium of a Transversely Vibrating Laminate

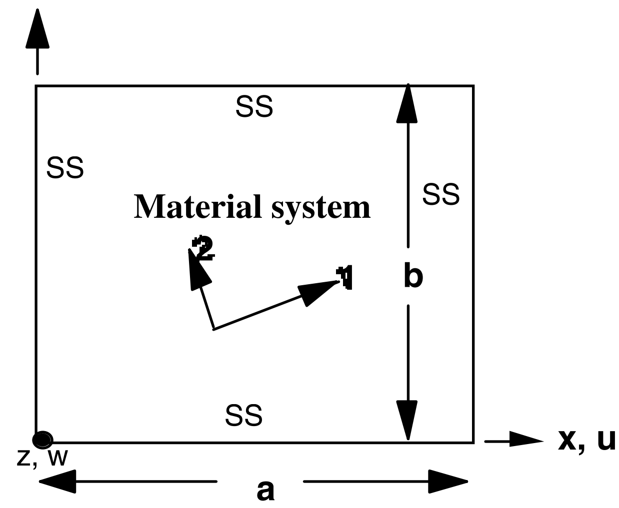
$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \rho h \ddot{w} = 0 \qquad (1)$$

Where $\ddot{w} = \partial^2 w / \partial t^2$, acceleration

4.4.2 Vibrations of a S-S laminated Plate.



Case (1): Symmetric Specially Orthotropic laminate

GDE: $D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} + \rho w_{,tt} = 0$ (2)

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = -D_{11}\ w_{,xx} - D_{12}\ w_{,yy} = 0$

@ y=0 & b: $w = 0$ & $M_y = -D_{12}\ w_{,xx} - D_{22}\ w_{,yy} = 0$

Selection of Displacement Functions: Free vibrations of an elastic continuum is harmonic, hence we can express the time variation of displacement in terms of sin, cos or $e^{i\omega t}$ functions. Using the principle of separation of variable, we can write the m & n th mode of vibration in the form of

$$w = a_{mn} e^{i\omega t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (3)$$

Where ω is the natural frequency of vibration, expressed in terms of radians/sec. Substituting for ŌwÕ in the GDE, for non-trivial solution, we get

$$\omega^2 = \frac{\pi^4}{\rho}\left\{D_{11}\left(\frac{m}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4\right\} \qquad (4)$$

ω is function of
(i) Elastic properties of the material.
(ii) Geometry (a and a/b)
(iii) Number of half-waves in x- and y-directions.

The fundamental or the lowest frequency is when m=n=1.

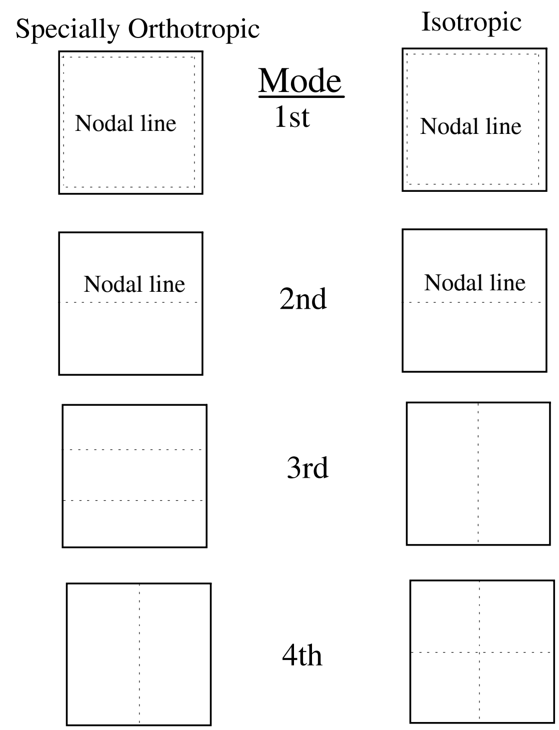
$$(\omega)_{Fundamental} = \frac{\pi^2}{b^2}\sqrt{\frac{D_{22}}{\rho}}\sqrt{\left\{\frac{D_{11}}{D_{22}}\left(\frac{b}{a}\right)^4 + 2\frac{(D_{12}+2D_{66})}{D_{22}}\left(\frac{b}{a}\right)^2 + 1\right\}} \qquad (5)$$

Example: For $D_{11}/D_{22} = 10$, $(D_{12} + 2D_{66})/D_{22} = 1$, and a/b=1 (Square Plate)

$$\omega = \frac{\pi^2}{b^2}\sqrt{\frac{D_{22}}{\rho}}\sqrt{10m^4 + 2m^2n^2 + n^4}$$

For isotropic material: $\omega = \frac{\pi^2}{b^2}\sqrt{\frac{D}{\rho}}\left(m^2 + n^2\right)$

	Specially orthotropic			Isotropic		
Mode	m	n	$\omega\left(\frac{b}{\pi}\right)^2\sqrt{\frac{\rho}{D_{22}}}$			$\omega\left(\frac{b}{\pi}\right)^2\sqrt{\frac{\rho}{D}}$
Ist	1	1	3.606	1	1	2.0
2nd	1	2	5.831	1	2	5.0
3rd	1	3	10.440	2	1	5.0
4th	2	1	13.000	2	2	8.0



Case 2: Symmetric Angle Ply Laminate

Because of symmetry, the inplane and transverse displacements GDE will decouple into 3 independent Equations. However bending-twisting coupling stiffness (D_{16} and D_{26}) are non zero.

GDE: $D_{11}w_{,xxxx} + 4D_{16}w_{,xxxy} + 2(D_{12} + 2D_{66})w_{,xxyy} + 4D_{26}w_{,xyyy} + D_{22}w_{,yyyy} + \rho w_{,tt} = 0$ (7)

Boundary Conditions:

@ x=0 & a: w = 0 & $M_x = -D_{11}\ w_{,xx} - D_{12}\ w_{,yy} - 2D_{16}\ w_{,xy} = 0$

@ y=0 & b: w = 0 & $M_y = -D_{12}\ w_{,xx} - D_{22}\ w_{,yy} - 2D_{26}\ w_{,xy} = 0$

Solution: Energy method is used to obtain the solution.

$\Pi = U + V + T$

Where T is the kinetic energy of the laminate, $T = \frac{1}{2}\int_V \rho \dot{w}^2 dV$

Assumed displacements:

$$w = a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{i\omega t}$$

Case 3: Anti-Symmetric Cross- Ply Laminate

Properties:

$A_{11} = A_{22}$ and $D_{11} = D_{22}$, and $B_{22} = -B_{11}$

Other non zero terms are: A_{12}, A_{66}, D_{12}, and D_{66}.

Normal-shear and bending-twisting terms are zero.

GDE:

$$A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} - B_{11}w_{,xxx} = 0 \quad (8)$$

$$(A_{12} + A_{66})u_{,xy} + A_{66}v_{,xx} + A_{11}v_{,yy} + B_{11}w_{,yyy} = 0 \quad (9)$$

$$D_{11}(w_{,xxxx} + w_{,yyyy}) + 2(D_{12} + 2D_{66})w_{,xxyy} - B_{11}(u_{,xxx} - u_{,yyy}) + \rho w_{,tt} = 0 \quad (10)$$

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = B_{11}u_{,x} - D_{11}w_{,xx} - D_{12}w_{,yy} = 0$

$v = 0$ & $N_x = A_{11}u_{,x} + A_{12}v_{,y} - B_{11}w_{,xx} = 0$

@ y=0 & b: $w = 0$ & $M_y = -B_{11}v_{,y} - D_{12}w_{,xx} - D_{22}w_{,yy} = 0$

$u = 0$ & $N_y = A_{12}u_{,x} + A_{22}v_{,y} + B_{11}w_{,yy} = 0$

Displacement Functions:

$$u = \bar{u}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t}$$

$$v = \bar{v}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t} \qquad (11)$$

$$w = \bar{w}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t}$$

Natural Frequency:

$$\omega^2 = \left(\frac{\pi^4}{\rho}\right)\left\{T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right\} \qquad (12)$$

Where

$$T_{11} = A_{11}\left(\frac{m}{a}\right)^2 + A_{66}\left(\frac{n}{b}\right)^2$$

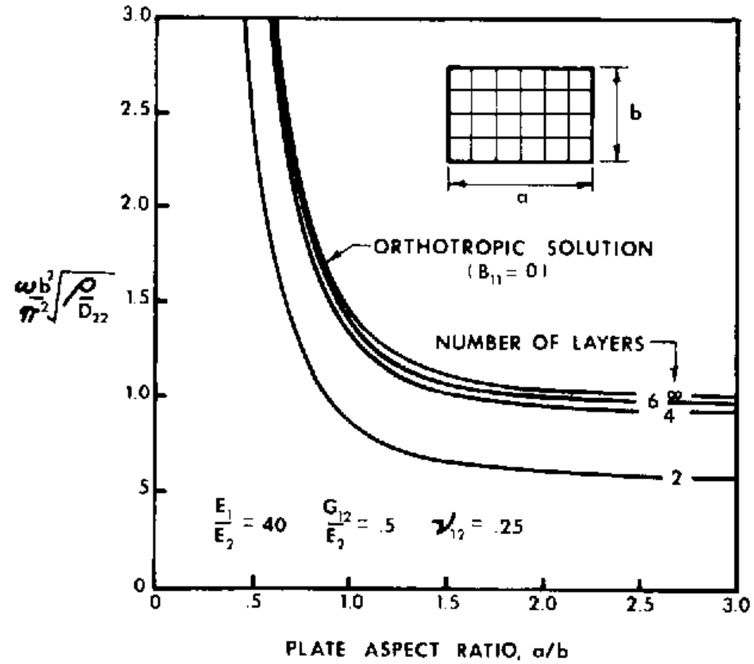
$$T_{12} = \left(A_{12} + A_{66}\right)\left(\frac{m}{a}\right)\left(\frac{n}{b}\right)$$

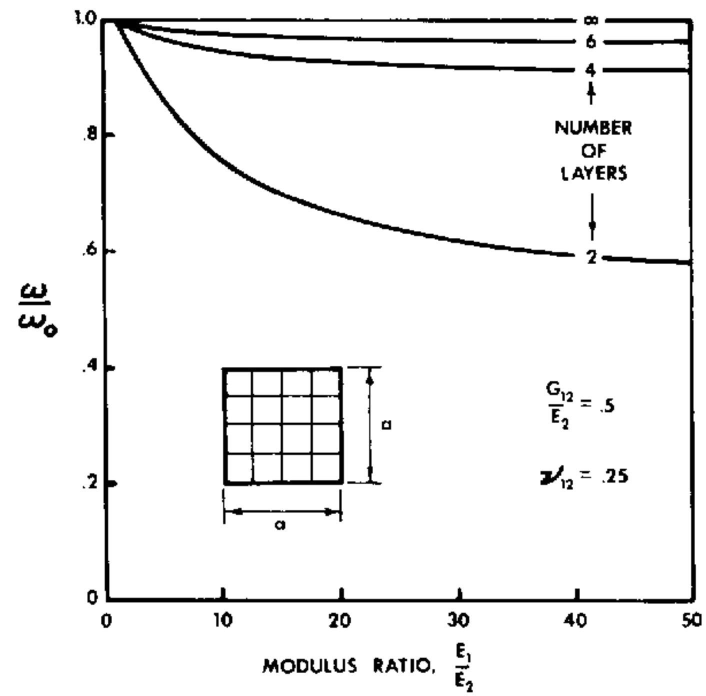
$$T_{13} = -B_{11}\left(\frac{m}{a}\right)^3$$

$$T_{22} = A_{11}\left(\frac{n}{b}\right)^2 + A_{66}\left(\frac{m}{a}\right)^2 \qquad (13)$$

$$T_{23} = B_{11}\left(\frac{n}{b}\right)^3$$

$$T_{33} = D_{11}\left\{\left(\frac{m}{a}\right)^4 + \left(\frac{n}{b}\right)^4\right\} + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2$$





Case 4: Anti-Symmetric Angle-Ply Laminate

Properties:

Non Zero Extensional Stiffness: A_{11}, A_{22}, A_{12} and A_{66}.
Non Zero Flexural Stiffness: D_{11}, D_{22}, D_{12}, and D_{66}.
Bending-Extensional Coupling Stiffness: B_{16} and B_{26}

GDE:

$$A_{11}u_{,xx} + A_{66}u_{,yy} + \left(A_{12} + A_{66}\right)v_{,xy} - 3B_{16}w_{,xxy} - B_{26}w_{,yyy} = 0 \tag{14}$$

$$\left(A_{12} + A_{66}\right)u_{,xy} + A_{66}v_{,xx} + A_{22}v_{,yy} - B_{16}w_{,xxx} - 3B_{26}w_{,xyy} = 0 \tag{15}$$

$$D_{11}w_{,xxxx} + 2\left(D_{12} + 2D_{66}\right)w_{,xxyy} + D_{22}w_{,yyyy} - B_{16}(3u_{,xxy} + v_{,xxx}) - B_{26}(u_{,yyy} + 3v_{,xyy}) + \rho w_{,tt} = 0 \tag{16}$$

Boundary Conditions:

@ x=0 & a: $w = 0$ & $M_x = B_{16}(u_{,y} + v_{,x}) - D_{11}w_{,xx} - D_{12}w_{,yy} = 0$
$u = 0$ & $N_{xy} = A_{66}(u_{,y} + v_{,x}) - B_{16}w_{,xx} - B_{26}w_{,yy} = 0$

@ y=0 & b: $w = 0$ & $M_y = B_{26}(u_{,y} + v_{,x}) - D_{12}w_{,xx} - D_{22}w_{,yy} = 0$
$v = 0$ & $N_{xy} = A_{66}(u_{,y} + v_{,x}) - B_{16}w_{,xx} - B_{26}w_{,yy} = 0$

Displacement Functions:

$$\begin{aligned} u &= \bar{u}\sin\tfrac{m\pi x}{a}\cos\tfrac{n\pi y}{b}e^{i\omega t} \\ v &= \bar{v}\cos\tfrac{m\pi x}{a}\sin\tfrac{n\pi y}{b}e^{i\omega t} \\ w &= \bar{w}\sin\tfrac{m\pi x}{a}\sin\tfrac{n\pi y}{b}e^{i\omega t} \end{aligned} \tag{17}$$

Natural Frequency:

$$\omega^2 = \left(\frac{\pi^4}{\rho}\right)\left\{T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right\} \qquad (18)$$

Where

$$T_{11} = A_{11}\left(\frac{m}{a}\right)^2 + A_{66}\left(\frac{n}{b}\right)^2$$

$$T_{12} = \left(A_{12} + A_{66}\right)\left(\frac{m}{a}\right)\left(\frac{n}{b}\right)$$

$$T_{13} = -\left\{3B_{16}\left(\frac{m}{a}\right)^2 + B_{26}\left(\frac{n}{b}\right)^2\right\}\left(\frac{n}{b}\right)$$

$$T_{22} = A_{22}\left(\frac{n}{b}\right)^2 + A_{66}\left(\frac{m}{a}\right)^2 \qquad (19)$$

$$T_{23} = -\left\{B_{16}\left(\frac{m}{a}\right)^2 + 3B_{26}\left(\frac{n}{b}\right)^2\right\}\left(\frac{m}{a}\right)$$

$$T_{33} = D_{11}\left(\frac{m}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4$$