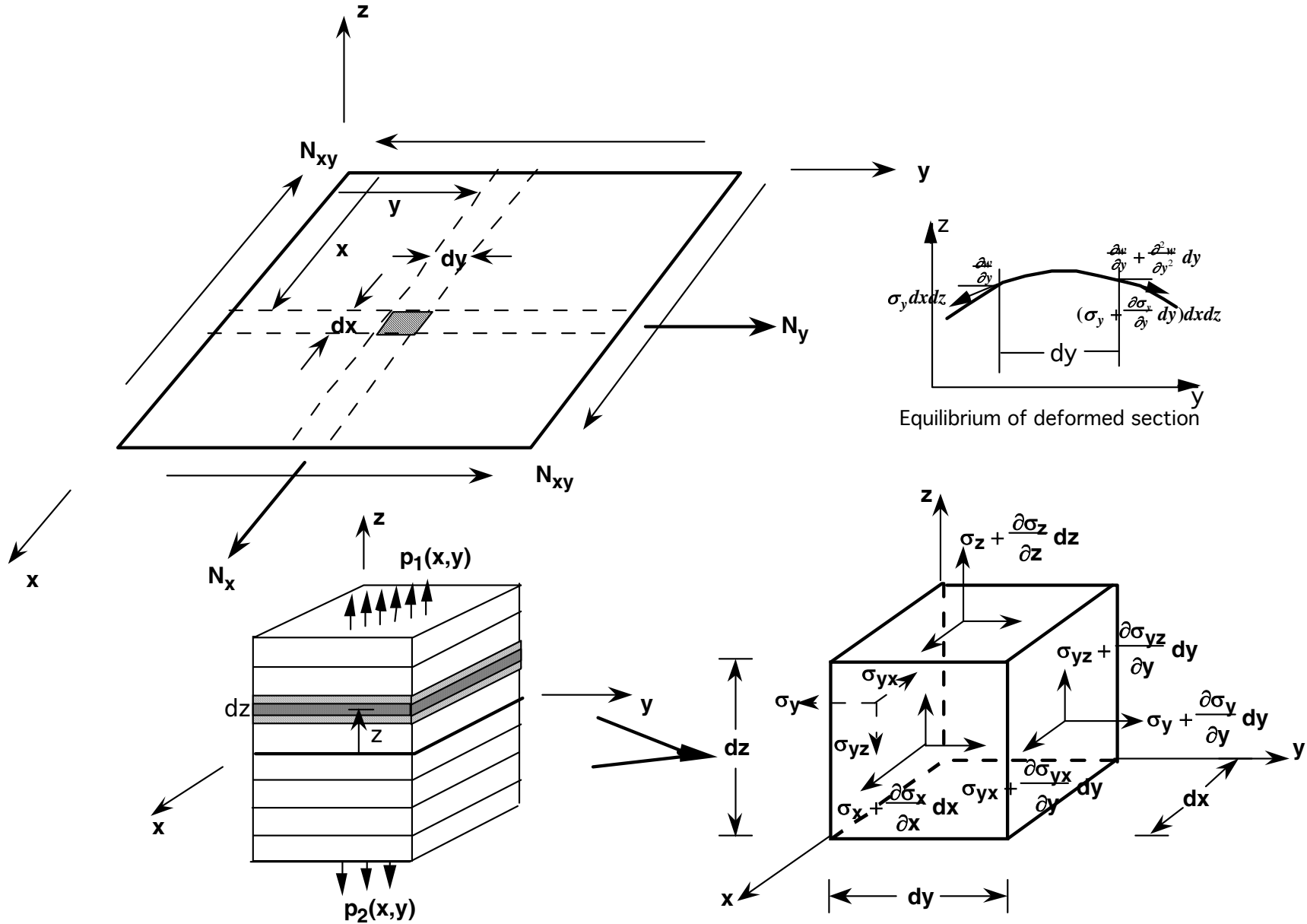


# Chapter 3: Governing Equations of Equilibrium of laminated Plates



**Applied Loading:**

1. Traction load  $p(x,y,t)$
2. Body force:  $F_x, F_y,$  and  $F_z$ ; Includes Inertial, Magnetic, etc.  $\rho \ddot{w}, \rho \dot{w},$  and  $\rho w$

**3.1 Equations of Equilibrium**

Consider the equilibrium of the elemental volume  $dx \times dy \times dz$  in y-direction

$$\left( \sigma_y + \frac{\partial \sigma_y}{\partial y} dy - \sigma_y \right) dx dz + \left( \sigma_{zy} + \frac{\partial \sigma_{zy}}{\partial z} dz - \sigma_{zy} \right) dx dy + \left( \sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} dx - \sigma_{xy} \right) dy dz + F_y dx dy dz = 0$$

Dividing throughout by  $dx \cdot dy \cdot dz$ , we get

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \frac{\partial \sigma_{xy}}{\partial x} + F_y = 0 \tag{2}$$

Similarly, the equilibrium in x-direction is

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x = 0 \tag{1}$$

A general equilibrium in z-direction includes the transverse force components due to in-plane stresses. These terms are a result of equilibrium of the deformed body and are needed for buckling and dynamic stability problems.

Consider the equilibrium in z-direction,

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z + \sigma_x \frac{\partial^2 w}{\partial x^2} + 2\sigma_{xy} \frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2} = 0 \tag{3}$$

### 3.2 Equations of Equilibrium in terms of Force Resultants

Integrate equations 1 to 3 with respect to z. For a laminate having 'n' number of layers,

$$\sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_x}{\partial x} \right)_k dz + \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_{yx}}{\partial y} \right)_k dz + \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_{zx}}{\partial z} \right)_k dz + \sum_{k=1}^n \int_{z_k}^{z_{k+1}} (F_x)_k dz = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + \bar{F}_x = 0 \quad (4)$$

Where,  $N_x$ , and  $N_{yx}$  are Inplane stress (Force resultants) and  $\bar{F}_x$  is the average body force per unit volume. Similarly, force resultant expression in y-direction is

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + \bar{F}_y = 0 \quad (5)$$

Integration of Equation 3, yields

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_1 - p_2 + \bar{F}_z + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (6)$$

Where  $p_1$  and  $p_2$  are the pressure loading on faces  $z=+h/2$  and  $-h/2$ . In laminated plate theory, we can replace the two pressures,  $p_1 - p_2 = p$ , resultant pressure loading.

### 3.3 Moment Resultants:

$$\sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_x}{\partial x} \right)_k z dz + \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_{yx}}{\partial y} \right)_k z dz + \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_{zx}}{\partial z} \right)_k z dz = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0 \quad (7)$$

Similarly,

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \quad (8)$$

Substituting Eqs. 7 and 8 in 6 results in a fundamental equations of motion of a laminated plate.

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p + \bar{F}_z + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (9)$$

Equations 1, 2, and 9 are general governing equations of motion of a laminated plate. These equation are valid for static, buckling, and dynamic (free vibration, dynamic response, and dynamic stability) problems. Depending on the type of problem, the equations simplify accordingly.

(a) **Static Stress Analysis**

$$\begin{aligned}\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0\end{aligned}\quad \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p + \bar{F}_z = 0$$

(b) **Buckling Analysis**

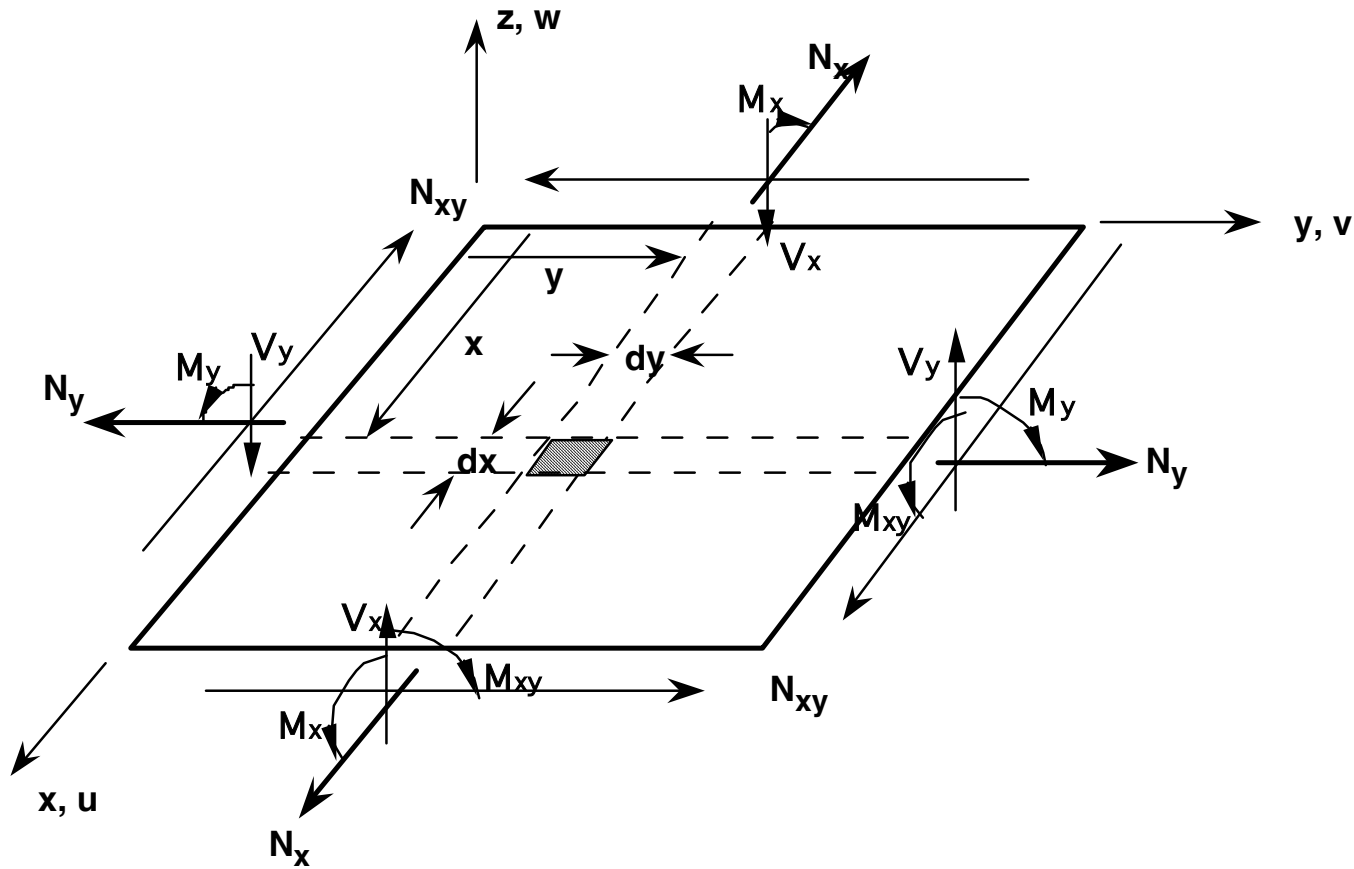
$$\begin{aligned}\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \bar{N}_x \frac{\partial^2 w}{\partial x^2} + \bar{N}_y \frac{\partial^2 w}{\partial y^2} + 2\bar{N}_{xy} \frac{\partial^2 w}{\partial x \partial y} &= 0\end{aligned}$$

Where  $\bar{N}_x$ ,  $\bar{N}_y$ , and  $\bar{N}_{xy}$  are the edge loads.

(c) **Free Vibration Analysis**

$$\begin{aligned}\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \rho h \ddot{w} &= 0\end{aligned}$$

### 3.4 Static Analysis



Governing equations of equilibrium

$$N_{x,x} + N_{yx,y} = 0 \quad (1) \quad N_{xy,x} + N_{y,y} = 0 \quad (2)$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} = -p \quad (3)$$

Substituting for force & moment results, we get for no thermal and moisture loading

$$\begin{aligned} A_{11}u_{,xx} + 2A_{16}u_{,xy} + A_{66}u_{,yy} + A_{16}v_{,xx} + (A_{12} + A_{66})v_{,xy} + A_{26}v_{,yy} \\ - B_{11}w_{,xxx} - 3B_{16}w_{,xxy} - (B_{12} + 2B_{66})w_{,xyy} - B_{26}w_{,yyy} = 0 \end{aligned} \quad (4)$$

$$\begin{aligned} A_{16}u_{,xx} + (A_{12} + A_{66})u_{,xy} + A_{26}u_{,yy} + A_{66}v_{,xx} + 2A_{16}v_{,xy} + A_{22}v_{,yy} \\ - B_{16}w_{,xxx} - (B_{12} + 2B_{66})w_{,xxy} - 3B_{26}w_{,xyy} - B_{22}w_{,yyy} = 0 \end{aligned} \quad (5)$$

$$\begin{aligned} D_{11}w_{,xxxx} + 4D_{16}w_{,xxxxy} + 2(D_{12} + 2D_{66})w_{,xxxyy} + 4D_{26}w_{,xyyyy} + D_{22}w_{,yyyyy} \\ - B_{11}u_{,xxx} - 3B_{16}u_{,xxy} - (B_{12} + 2B_{66})u_{,xyy} - B_{26}u_{,yyy} \\ - B_{16}v_{,xxx} - (B_{12} + 2B_{66})v_{,xxy} - 3B_{26}v_{,xyy} - B_{22}v_{,yyy} = p \end{aligned} \quad (6)$$

Equations 4 to 6 represent the general governing differential equations of equilibrium of a statically loaded anisotropic laminate. The equations involve coupled differentials of u, v, and w displacements. The equations significantly simplifies for a symmetric laminate ( $B_{ij} = 0$ ).

### Equations of Equilibrium for Symmetric laminates:

$$A_{11}u_{,xx} + 2A_{16}u_{,xy} + A_{66}u_{,yy} + A_{16}v_{,xx} + (A_{12} + A_{66})v_{,xy} + A_{26}v_{,yy} = 0 \quad (7)$$

$$A_{16}u_{,xx} + (A_{12} + A_{66})u_{,xy} + A_{26}u_{,yy} + A_{66}v_{,xx} + 2A_{16}v_{,xy} + A_{22}v_{,yy} = 0 \quad (8)$$

$$D_{11}w_{,xxxx} + 4D_{16}w_{,xxxxy} + 2(D_{12} + 2D_{66})w_{,xxxyy} + 4D_{26}w_{,xyyyy} + D_{22}w_{,yyyyy} = p \quad (9)$$

### **Kirchhoff Shear (V)**

Moment equilibrium equation:

$$M_{x,x} + M_{xy,y} - Q_x = 0$$

$$M_{x,x} - V_x = 0 \quad \text{or} \quad V_x = Q_x - M_{xy,y} \quad (10)$$

$V_x$  is called the Kirchhoff shear force.

## **5.2 Boundary Conditions**

General Curved Boundary Condition:

### **Analysis Methods:**

- Classical Methods
- Energy Methods
- Numerical Methods (Finite Element and Boundary Element Methods)

## 4 ANALYSIS OF LAMINATES

### 4.1 STATIC ANALYSIS

#### 4.1.1 Energy Method of Analysis of Structures

##### Principle of Total Potential Energy:

If the deformation of an elastic body can be expressed in terms of admissible functions that satisfy the kinematic boundary conditions, then the total potential energy (TPE) of the body is stationary. In other words, the first variation of the TPE with respect to coefficients of the admissible functions is zero. These set of equations satisfy global equations of equilibrium.

$$\text{Total potential energy, } \Pi = U + V = \int_V \int_0^\varepsilon \sigma d\varepsilon dv - \int p du$$

where  $U$  = Strain energy of the elastic body  
 $V$  = Potential energy due to work done by the applied forces  
 $\sigma$ ,  $\varepsilon$ ,  $u$ , and  $p$  are stress, strain, displacement, and the loading, respectively.

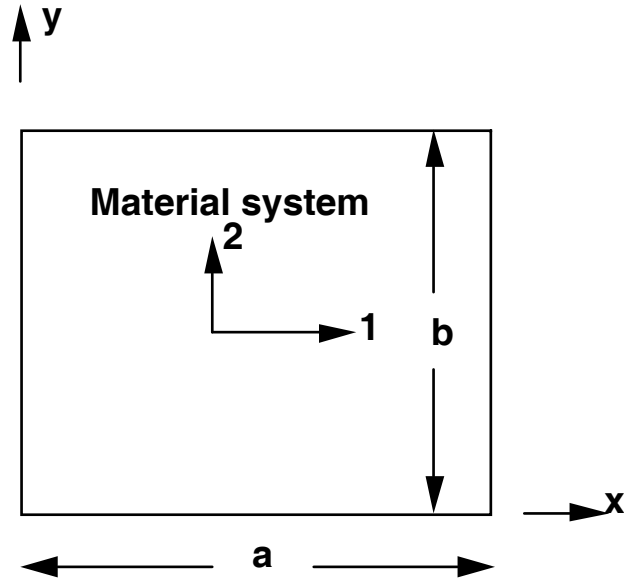
Principle:  $\frac{\partial \Pi}{\partial c_i} = 0$  ;  $C_i$  represent the generalized coordinates.

##### Procedure:

- Step 1: Select admissible functions that satisfy the geometric boundary conditions of the problem. Define the displacements as a function of admissible functions and undetermined constants called generalized displacements.  $u = c_i \phi_i$ .
- Step 2: Calculate the total potential energy ( $\Pi = U + V$ ) of the body using the appropriate **strain-displacement and constitutive equations**.

- Step 3: Minimize the TPE ( $\Pi$ ) with respect to the generalized displacements. This results in a set of algebraic equations to be solved for generalized constants.
- Step 4: Number of equations will be equal to number of constants. Then Substitute the constants in the displacement equation. This completes the analysis.

**Example:** Clamped-clamped rectangular plate subjected to an uniformly distributed load,  $-q$ .



**Boundary Conditions:**

$$w = 0; \text{ and } \frac{\partial w}{\partial x} = 0 \text{ @ } x = 0 \text{ and } x = a$$

$$w = 0; \text{ and } \frac{\partial w}{\partial y} = 0 \text{ @ } y = 0 \text{ and } y = b$$

**Displacement approximation:**

$$w(x,y) = \prod_{i=1}^m \prod_{j=1}^n c_{ij} \left(1 - \cos \frac{2i\pi x}{a}\right) \left(1 - \cos \frac{2j\pi y}{b}\right)$$

**One-term approximation:**  $w(x,y) = c \left(1 - \cos \frac{2\pi x}{a}\right) \left(1 - \cos \frac{2\pi y}{b}\right)$

**Total Potential Energy,  $\Pi$ :** (Specially orthotropic symmetric laminate)

$$\Pi(w) = \int_0^b \int_0^a \left\{ \frac{1}{2} \left[ \mathbf{D}_{11} \left( \frac{\check{z}^2 w}{\check{z}x^2} \right)^2 + 2\mathbf{D}_{12} \left( \frac{\check{z}^2 w}{\check{z}x^2} \right) \left( \frac{\check{z}^2 w}{\check{z}y^2} \right) + \mathbf{D}_{22} \left( \frac{\check{z}^2 w}{\check{z}y^2} \right)^2 + 4\mathbf{D}_{66} \left( \frac{\check{z}^2 w}{\check{z}x \check{z}y} \right)^2 \right] - qw \right\} dx dy$$

Substituting for  $w$  in the above equation, and performing partial differentiation w.r.to  $c$  leads to

$$4\mathbf{a}\mathbf{b}\pi^4 c \left[ \frac{3\mathbf{D}_{11}}{\mathbf{a}^4} + \frac{2\mathbf{D}_{12}}{\mathbf{a}^2\mathbf{b}^2} + \frac{3\mathbf{D}_{22}}{\mathbf{b}^4} + \frac{4\mathbf{D}_{66}}{\mathbf{a}^2\mathbf{b}^2} \right] = q\mathbf{a}\mathbf{b}$$

**Example:**

For a square ( $\mathbf{a} = \mathbf{b}$ ) and isotropic plate  $\{\mathbf{D}_{11} = \mathbf{D}_{22} = \mathbf{D}$ ,  $\mathbf{D}_{12} = \nu\mathbf{D}$ , and  $\mathbf{D}_{66} = (1-\nu)\mathbf{D}/2\}$ , the expression for  $c$  simplifies to

$$c = \frac{q\mathbf{a}^4}{32\pi^4\mathbf{D}}$$

therefore, the expression for  $w$  is

$$w(x,y) = \frac{q\mathbf{a}^4}{32\pi^4\mathbf{D}} \left( 1 - \cos \frac{2\pi x}{\mathbf{a}} \right) \left( 1 - \cos \frac{2\pi y}{\mathbf{b}} \right)$$

$w_{\max}$  is at the center of the plate,

$$w_{\max} = \frac{q\mathbf{a}^4}{8\pi^4\mathbf{D}} = 0.00128 \frac{q\mathbf{a}^4}{\mathbf{D}}$$

**Series solution by Evans (1939):**

$w_{\max} = 0.00126 \frac{qa^4}{D}$ ; The approximate solution is about 1% larger than the exact solution.