Chapter 2: Hygrothermal Analysis Of Laminates

2.1 Introduction

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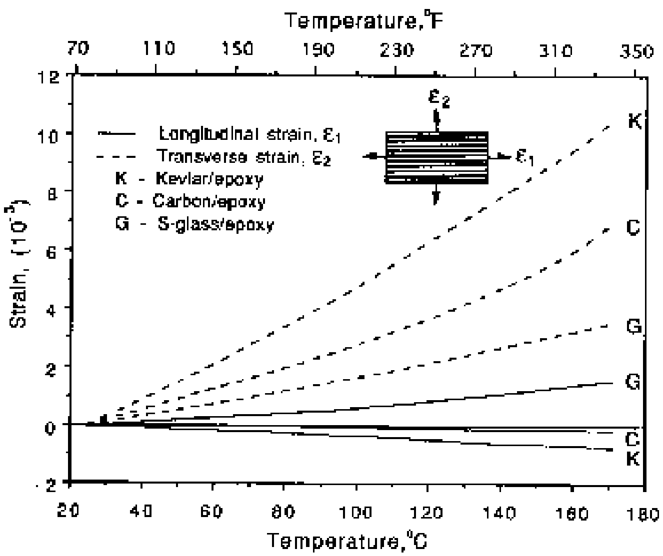
Thermal Expansion Properties of Composites

Variation of Thermal Expansion Coefficients with Temperature



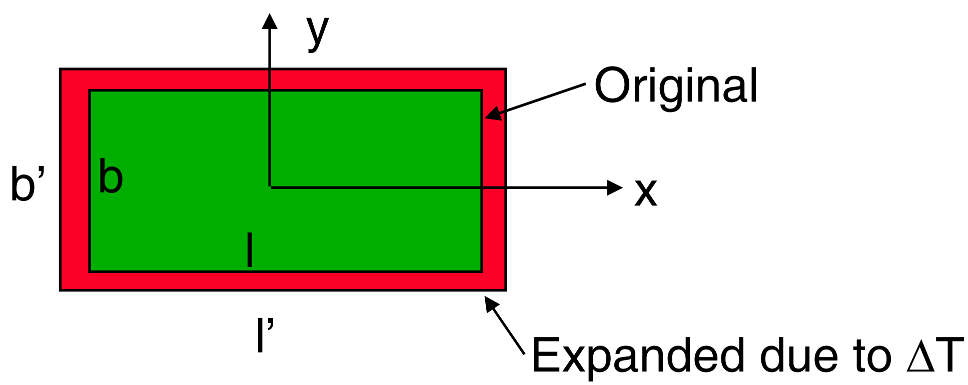
Coefficients of Thermal Expansion

Material	Longitudinal coefficient of thermal expansion, α_1 10^{-6}/°C (10^{-6}/°F)				Transverse coefficient of thermal expansion, α_2 10^{-6}/°C (10^{-6}/°F)			
	24°C	(75°F)	177°C	(350°F)	24°C	(75°F)	177°C	(350°F)
Boron/epoxy (boron/AVCO 5505)	6.1	(3.4)	6.1	(3.4)	30.3	(16.9)	37.8	(21.0)
Boron/polyimide (boron/WRD 9371)	4.9	(2.7)	4.9	(2.7)	28.4	(15.8)	28.4	(15.8)
Carbon/epoxy (AS4/3501-6)	−0.9	(−0.5)	−0.9	(−0.5)	27.0	(15.0)	34.2	(19.0)
Carbon/polyimide (modmor I/WRD 9371)	0.4	(−0.2)	−0.4	(−0.2)	25.3	(14.1)	25.3	(14.1)
S glass/epoxy (Scotchply 1009-26-5901)	3.8	(2.1)	3.8	(2.1)	16.7	(9.3)	54.9	(30.5)
S-glass/epoxy (S-glass/ERLA 4617)	6.6	(3.7)	14.1	(7.9)	19.7	(10.9)	26.5	(14.7)
Kevlar/epoxy (Kevlar 49/ERLA 4617)	4.0	(−2.2)	−5.7	(−3.2)	57.6	(32.0)	82.8	(46.0)

2.2 Hygrothermal Coefficients of a Lamina

6.2.1 Coefficients of Thermal Expansion

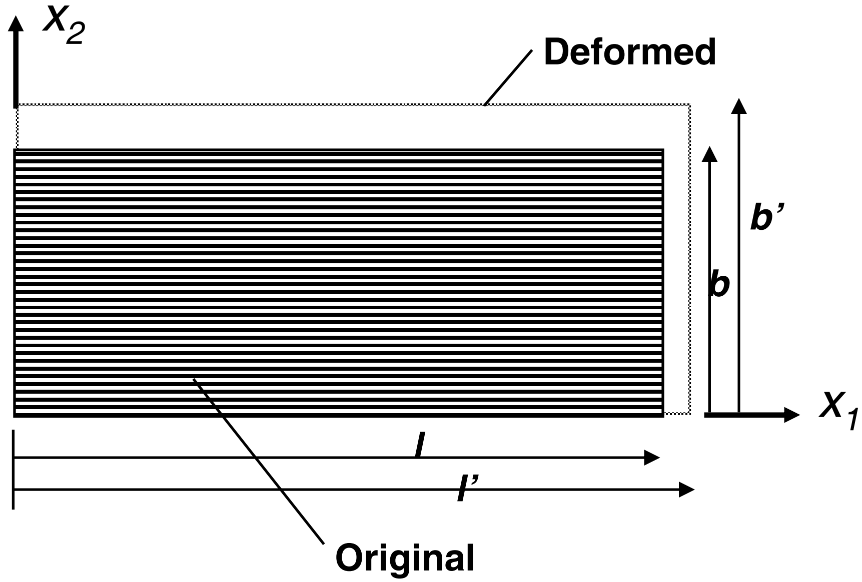
(a) Isotropic Materials



Coefficient of thermal expansion, $\alpha_x^T = \alpha_y^T = \alpha^T = \frac{l' - l}{l \Delta T}$

Units: in/in/^{o}F or m/m /^{o}C

(B) Orthotropic Materials



Coefficient of thermal expansion

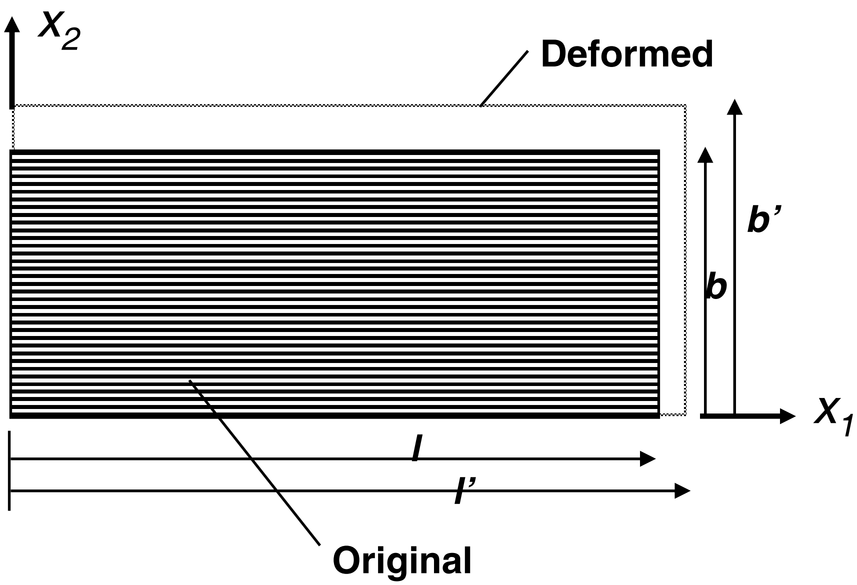
In x_1-direction $\alpha_1^T = \frac{l'-l}{l\Delta T}$

In x_2-direction $\alpha_2^T = \frac{b'-b}{b\Delta T}$

Thermal strains: $\{\varepsilon\} = \begin{Bmatrix} \alpha_1^T \\ \alpha_2^T \\ 0 \end{Bmatrix} \Delta T$

2.2.2 Coefficients of Moisture Expansion

All organic composites absorbs moisture. The absorption depends on the relative humidity to which it is exposed and its moisture content. For a given RH, temperature, and atmospheric pressure composite will have a sturation value. This is moisture content that the material will reach, if it is exposed for a very long time. This is a fixed value for a material. The moisture content is expressed as percent change in weight of the material. Like thermal expansion,increase in moisture would also expands the material. The orthotropic materials have two coefficients of moisture expansion, one along the fiber and the other across the fiber.



Change in moisture ΔM

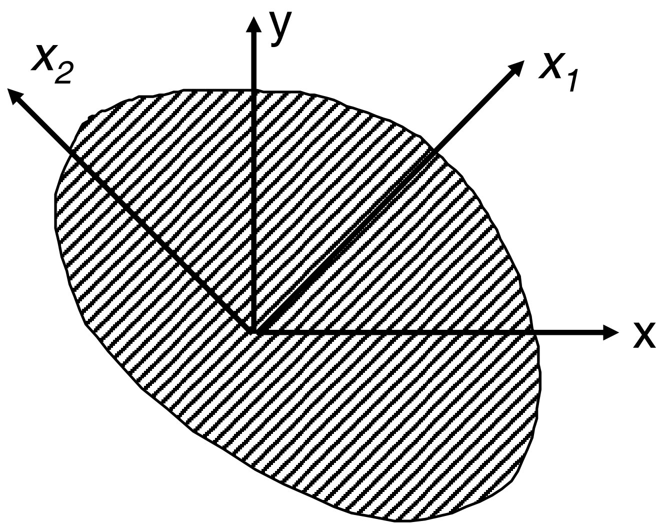
Coefficient of moisture expansion

In x_1-direction $\beta_1^T = \frac{l' - l}{l \Delta M}$

In x_2-direction $\beta_2^T = \frac{b' - b}{b \Delta M}$

Moisture strains: $\{\varepsilon^M\} = \begin{Bmatrix} \beta_1^M \\ \beta_2^M \\ 0 \end{Bmatrix} \Delta M$

2.3 Coefficients of Thermal & Moisture Expansion for Lamina in Arbitrary Orientation



Recall the strain transformation:

$$\{\varepsilon^T\}_{xy} = [T_\varepsilon]\{\varepsilon^T\}_{1-2}$$

Where

$$[T_\varepsilon] = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix}$$

m=cosθ and n=sinθ

Thermal strains in *x-y* due to *ΔT* are:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \alpha_1^T \\ \alpha_2^T \\ 0 \end{Bmatrix} \Delta T \quad = \begin{Bmatrix} \alpha_x^T \\ \alpha_y^T \\ \alpha_{xy}^T \end{Bmatrix}$$

Coefficients of thermal expansion in *x-y*:

$$\alpha_x^T = m^2\alpha_1^T + n^2\alpha_2^T \qquad \alpha_y^T = n^2\alpha_1^T + m^2\alpha_2^T \qquad \alpha_{xy}^T = 2mn(\alpha_1^T - \alpha_2^T)$$

Thermal strain in the kth layer is

$$\begin{Bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{Bmatrix}_k = \Delta T_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k$$

where,

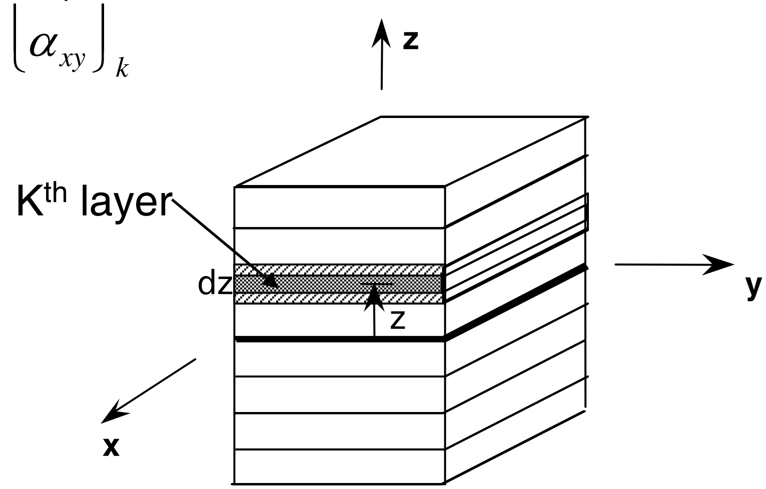
$$\alpha_x = \alpha_1 \cos^2 \theta_k + \alpha_2 \sin^2 \theta_k$$
$$\alpha_y = \alpha_1 \sin^2 \theta_k + \alpha_2 \cos^2 \theta_k$$
$$\alpha_{xy} = 2(\alpha_1 - \alpha_2)\cos\theta_k \sin\theta_k$$

Thermal stresses in the kth layer

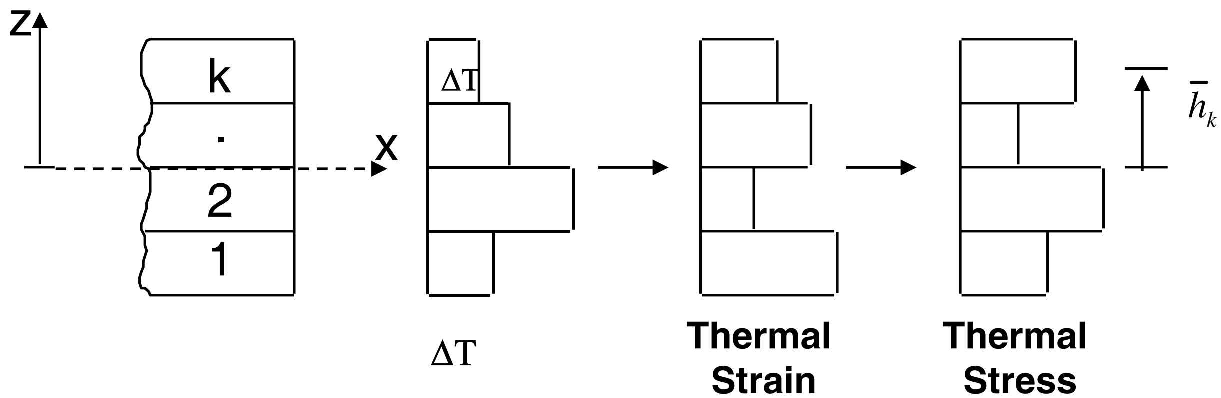
$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix}_k = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{Bmatrix} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_{xy} \Delta T \end{Bmatrix}_k = [Q]_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k \Delta T_k$$

Where

$$\begin{Bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{Bmatrix} = \Delta T_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k$$



Thermal strain & stress distribution through the thickness



Summary:

Coefficients of thermal expansion in *x-y*:

$$\alpha_x^T = m^2\alpha_1^T + n^2\alpha_2^T$$

$$\alpha_y^T = n^2\alpha_1^T + m^2\alpha_2^T$$

$$\alpha_{xy}^T = 2mn(\alpha_1^T - \alpha_2^T)$$

Coefficients of moisture expansion in *x-y*:

$$\beta_x^M = m^2\beta_1^M + n^2\beta_2^M$$

$$\beta_y^M = n^2\beta_1^M + m^2\beta_2^M$$

$$\beta_{xy}^M = 2mn(\beta_1^M - \beta_2^M)$$

2.4 Hygrothermal Stress Resultants

2.4.1 Thermal force resultants [N^T] : Defined as the integral of thermal stresses through the thickness of the laminate per unit width.

$$\{N^T\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma^T\} dz$$

Substituting for [σ^T] and integrating with respect to z, we get

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix}_k = \sum_{k=1}^{N} \Delta T_k \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k t_k$$

Where $t_k = h_{k+1} - h_k$

For linearly varying temperatures through lamina thickness ΔT_k can be computed at mid-thickness of the k^{th} layer. K varies from 1 through N, number of layers.

2.4.2 Thermal Moment Resultant

For constant ΔT_k in the k^{th} layer

$$\{M^T\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma^T\} z dz = \sum_{k=1}^{N} \{\sigma^T\} t_k \bar{h}_k$$

Where $\bar{h}_k = \dfrac{h_{k+1} + h_k}{2}$

$$\begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix}_k = \sum_{k=1}^{N} \Delta T_k \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k t_k \bar{h}_k$$

Following the same procedure as the thermal distribution, we can show that

2.4.3 Moisture Force resultant *[N^m]* is

$$\begin{Bmatrix} N_x^M \\ N_y^M \\ N_{xy} \end{Bmatrix}_k = \sum_{k=1}^{N} \Delta M_k \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{Bmatrix}_k t_k$$

Where

$$\beta_{x_k} = \beta_1 \cos^2\theta_k + \beta_2 \sin^2\theta_k$$

$$\beta_{y_k} = \beta_1 \sin^2\theta_k + \beta_2 \cos^2\theta_k$$

$$\beta_{xy_k} = 2(\beta_1 - \beta_2)\cos\theta_k \sin\theta_k$$

2.4.4 Moisture stress resultant is

$$\begin{Bmatrix} M_x^M \\ M_y^M \\ M_{xy} \end{Bmatrix}_k = \sum_{k=1}^{N} \Delta M_k \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{Bmatrix}_k t_k \bar{h}_k$$

2.5 Concept of Total Strain and Mechanical Strain

<u>Total strain</u>: Externally observed deformation is called the total strain. It is calculated from the derivatives of the deformations u*, v, & w*.

The total strain $\{\varepsilon^t\}$ **is**

$$\begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_x^t \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial u_o}{\partial y} \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

Or

$$\{\varepsilon^t\} = \{\varepsilon^m\} + \{\varepsilon^T\} + \{\varepsilon^M\}$$

$\{\varepsilon^m\}$ **is the strain caused by applied loads and the boundary restraints. This is called the mechanical strain**

$$\{\varepsilon^m\} = \{\varepsilon^t\} - \{\varepsilon^T\} - \{\varepsilon^M\}$$

2.6 Mechanical & Hygrothermal stress resultants

Mechanical stress $\{\sigma^m\}$ is calculated from the usual Stress-Strain relation as

$$\{\sigma^m\} = [Q_{xy}]\{\varepsilon^t\} - [Q_{xy}]\{\varepsilon^T\} - [Q_{xy}]\{\varepsilon^M\}$$

Or

$$\{\sigma^m\} = \{\sigma^t\} - \{\sigma^T\} - \{\sigma^M\}$$

The total stress is:

$$\{\sigma^m\} + \{\sigma^T\} + \{\sigma^M\} = \{\sigma^t\}$$

Associated Force resultants are:

$$\{N\} + \{N^T\} + \{N^M\} = \{N^t\} = \{\overline{N}\}$$

Associated Moment resultants are:

$$\{M\} + \{M^T\} + \{M^M\} = \{M^t\} = \{\overline{M}\}$$

2.7 Hygrothermoealstic Force-Displacement Relations

Force resultant equation:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + \begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} + \begin{Bmatrix} N_x^M \\ N_y^M \\ N_{xy}^M \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} \\ A_{yx} & A_{yy} & A_{ys} \\ A_{sx} & A_{sy} & A_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_s^t \end{Bmatrix} + \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{yx} & B_{yy} & B_{ys} \\ B_{sx} & B_{sy} & B_{ss} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

Moment resultant equation:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} + \begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} + \begin{Bmatrix} M_x^M \\ M_y^M \\ M_{xy}^M \end{Bmatrix} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{yx} & B_{yy} & B_{ys} \\ B_{sx} & B_{sy} & B_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_s^t \end{Bmatrix} + \begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{yx} & D_{yy} & D_{ys} \\ D_{sx} & D_{sy} & D_{ss} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

Note superscript for mechanical stress resultant is dropped.

$$\{\overline{\mathrm{N}}\} = \{\mathrm{N}\} + \{\mathrm{N}^{\mathrm{T}}\} + \{\mathrm{N}^{\mathrm{M}}\} = [\mathrm{A}]\{\varepsilon^{\mathrm{t}}\} + [\mathrm{B}]\{\kappa^{\mathrm{t}}\}$$

$$\{\overline{\mathrm{M}}\} = \{\mathrm{M}\} + \{\mathrm{M}^{\mathrm{T}}\} + \{\mathrm{M}^{\mathrm{M}}\} = [\mathrm{B}]\{\varepsilon^{\mathrm{t}}\} + [\mathrm{D}]\{\kappa^{\mathrm{t}}\}$$

Stress Resultants & Strain Eq.

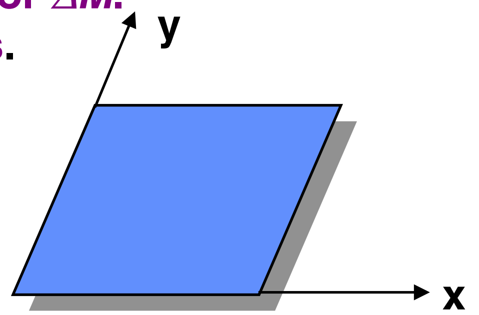
$$\begin{Bmatrix} \overline{N} \\ \overline{M} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^t \\ \kappa^t \end{Bmatrix}$$

Strain & Stress Resultants Eq.

$$\begin{Bmatrix} \varepsilon^t \\ \kappa^t \end{Bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{Bmatrix} \overline{N} \\ \overline{M} \end{Bmatrix}$$

2.8 Laminate Coefficient of Thermal and Moisture Expansions

Consider a multidirectional laminate with no restraint on the boundary subjected to change in temperature and moisture.
The mechanical load is zero.
Therefore the over all deformation is only due to *ΔT* and/or *ΔM*.
The laminate expansion is related to its stress resultants.



No boundary restraint

Consider $\Delta T=1$ and $\Delta M=0$.
Total strains are related to laminate thermal expansion coefficients.

$$\begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_{xy}^t \end{Bmatrix} = \Delta T \begin{Bmatrix} \bar{\alpha}_x \\ \bar{\alpha}_y \\ \bar{\alpha}_{xy} \end{Bmatrix} = \begin{Bmatrix} \bar{\alpha}_x \\ \bar{\alpha}_y \\ \bar{\alpha}_{xy} \end{Bmatrix}$$

Where $\bar{\alpha}_y$ $\bar{\alpha}_x$ $\bar{\alpha}_{xy}$ are the laminate CTEs to be determined.

Strain - Stress resultant equation is

$$\{\varepsilon^t\} = [a]\{\bar{N}\} + [b]\{\bar{M}\}$$

Where $\{\bar{N}\} = \{N\} + \{N^T\} + \{N^M\} = \{N^T\}$

Similarly

$\{\bar{M}\} = \{M^T\}$

Therefore, the expressions for CTE are:

$$\begin{Bmatrix} \bar{\alpha}_x \\ \bar{\alpha}_y \\ \bar{\alpha}_{xy} \end{Bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xs} \\ a_{yx} & a_{yy} & a_{ys} \\ a_{sx} & a_{sy} & a_{ss} \end{bmatrix} \begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} + \begin{bmatrix} b_{xx} & b_{xy} & b_{xs} \\ b_{yx} & b_{yy} & b_{ys} \\ b_{sx} & b_{sy} & b_{ss} \end{bmatrix} \begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix}$$

For a symmetric laminate: $\{\bar{\alpha}_{xy}\} = [a]\{N^T\}$

Similarly, we can derive the laminate coefficients of moisture expansion (CME) as:

$$\begin{Bmatrix} \bar{\beta}_x \\ \bar{\beta}_y \\ \bar{\beta}_{xy} \end{Bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xs} \\ a_{yx} & a_{yy} & a_{ys} \\ a_{sx} & a_{sy} & a_{ss} \end{bmatrix} \begin{Bmatrix} N_x^M \\ N_y^M \\ N_{xy}^M \end{Bmatrix} + \begin{bmatrix} b_{xx} & b_{xy} & b_{xs} \\ b_{yx} & b_{yy} & b_{ys} \\ b_{sx} & b_{sy} & b_{ss} \end{bmatrix} \begin{Bmatrix} M_x^M \\ M_y^M \\ M_{xy}^M \end{Bmatrix}$$

For a symmetric laminate: $\{\bar{\beta}_{\mathrm{xy}}\} = [\mathrm{a}]\{\mathrm{N}^{\mathrm{T}}\}$

2.9 Warpage:

Warpage is an out-of-plane deformation in asymmetric laminates due to hygrothermal loading. Bending asymmetries result from an asymmetric lay-up and thermal and moisture gradients through-the-thickness. The warpage can be calculated from the strain-Stress resultant equations

$$\{\kappa\}_{xy} = [c]\{N^{HT}\} + [d]\{M^{HT}\}$$

$$\{\kappa\}_{xy} = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

Integration of this equation yields:

$$w = -1/2(\kappa_x x^2 + \kappa_y y^2 + \kappa_{xy} xy) + Rigid\ body\ motion$$

2.10 Residual Strains & Stresses:

Because of lamination process, the ply free expansion strains are different from the total laminate strains. Therefore the individual plies develop residual strains and stresses.

Procedure:

1. Calculate the laminate total strains from the hygrothermal stress resultants.
2. Consider kth layer,

Residual strain = Laminate strain - Free Expansion Strain

Residual stress = Stiffness x Residual strain.

$$\begin{Bmatrix} \varepsilon^t \\ --- \\ \kappa \end{Bmatrix} = \begin{bmatrix} a & \vdots & b \\ -- & + & -- \\ c & \vdots & d \end{bmatrix} \begin{Bmatrix} \overline{N} \\ --- \\ \overline{M} \end{Bmatrix}$$

$$\left\{\varepsilon^r\right\}_{residual} = \left\{\varepsilon^t\right\} + z\left\{\kappa\right\} - \left\{\varepsilon^T\right\}_z$$

$$\left\{\sigma^m\right\}_{residual} = \left[Q_{xy}\right]\left\{\varepsilon^r\right\}_{residual}$$ **@ that z.**

HW: 6.5 to 6.24

Laminate Cases

Characteristics of A, B & D for various types of laminates have been studied.
The cases of interest are:

(a) Symmetric laminate: cross-ply and angle-ply laminates.
(b) Residual stresses
(c) Warpage

Ex: Symmetric laminate

B-matrix is zero.
For constant temperature though-the-thickness

$$\left\{M^T\right\}_k = \sum_{k=1}^{N} \Delta T_k \left[Q_{xy}\right]_k \left\{\alpha_{xy}\right\}_k t_k \bar{h}_k = 0$$

$$\{N\} + \left\{N^T\right\} + \left\{N^M\right\} = \left[A_{xy}\right]\left\{\varepsilon^t\right\}$$

$$\{M\} = \left[D_{xy}\right]\left\{\kappa^t\right\}$$

For Cross-ply:

$$A_{xs} = A_{ys} = D_{xs} = D_{ys} = 0$$