

Chapter 1: Review of Classical Laminate Theory

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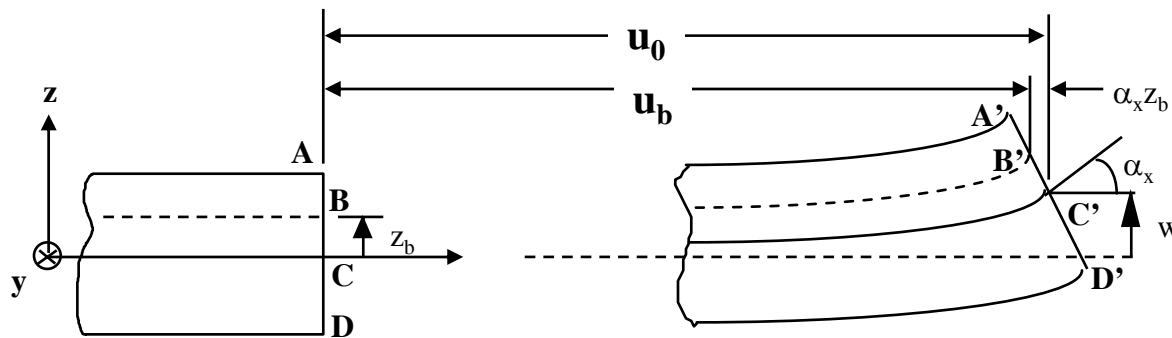
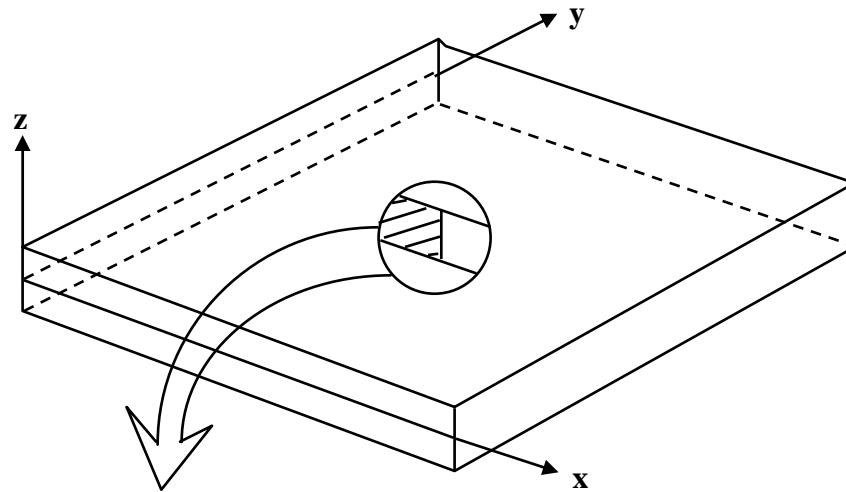
Single layer

Symmetric laminates

Balanced laminates

Quasi-isotropic laminates

Laminate and Deformation Parameters



1.1 Love-Kirchhoff Assumptions

1. Each layer of the laminate is quasihomogeneous and orthotropic.
2. The laminate is thin compared to the lateral dimensions and is loaded in its plane. Plane stress state.
3. All displacements are small compared to the laminate thickness.
4. Displacements are continuous throughout the laminate.
5. Straight lines normal to the middle surface remain straight and normal to that surface after deformation.
 - Inplane displacements vary linearly through the thickness,
 - Transverse shear strains (γ_{xz} & γ_{yz}) are negligible.
6. Transverse normal strain ε_z is negligible compared to the in-plane strains ε_x and ε_y .
7. Strain-displacement and stress-strain relations are linear.

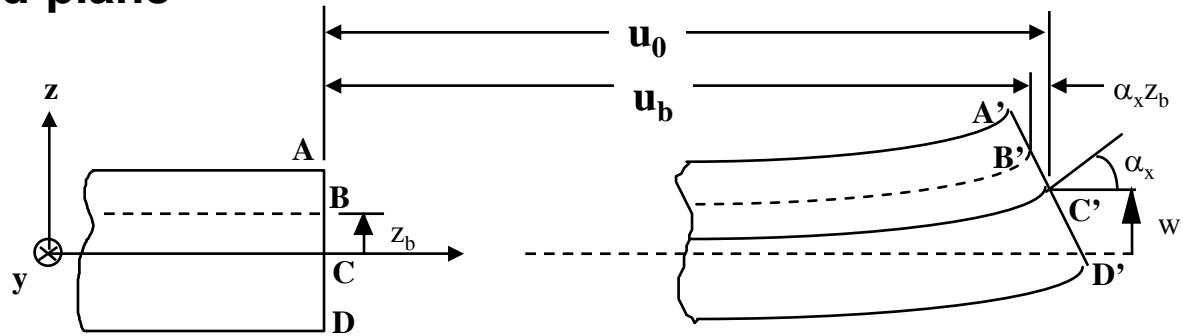
1.2 Strain-Displacement Relations

Displacements @ mid-plane

$$u_0 = u_0(x, y)$$

$$v_0 = v_0(x, y)$$

$$w = f(x, y)$$



From assumption 5 rotation of normal to the mid planes are

$$\alpha_x = \frac{\partial w}{\partial x}$$

$$\alpha_y = \frac{\partial w}{\partial y}$$

In-plane displacements @ B are:

$$u = u_0 - z \frac{\partial w}{\partial x}$$

$$v = v_0 - z \frac{\partial w}{\partial y}$$

From assumption 6, $w(x, y, z) = w(x, y)$

Using the small deformation linear theory (Assumption 3 & 7), strain-displacement equations can be written as:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \gamma_s = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y}$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

Where the mid-plane strains are given by:

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y}$$

$$\gamma_{xy}^0 = \gamma_s^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$

Curvatures are:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}$$

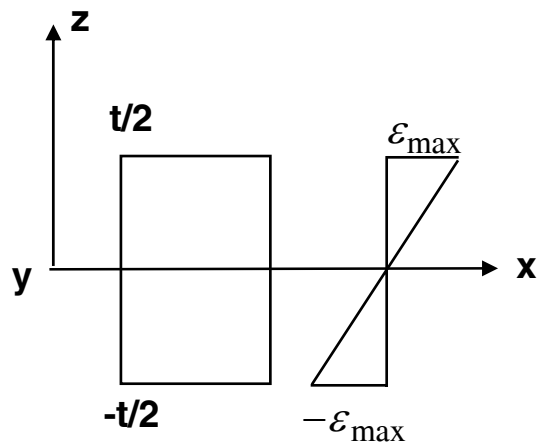
$$\kappa_y = -\frac{\partial^2 w}{\partial y^2}$$

$$\kappa_{xy} = \kappa_s = -\frac{2\partial^2 w}{\partial x \partial y}$$

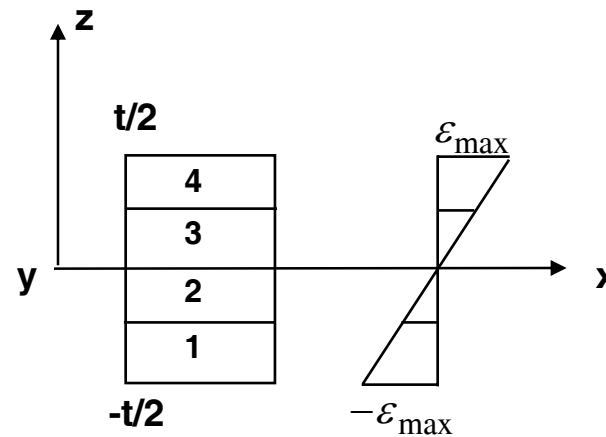
Strains at any point (x,y,z) are:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{bmatrix} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}$$

Isotropic/Orthotropic Material

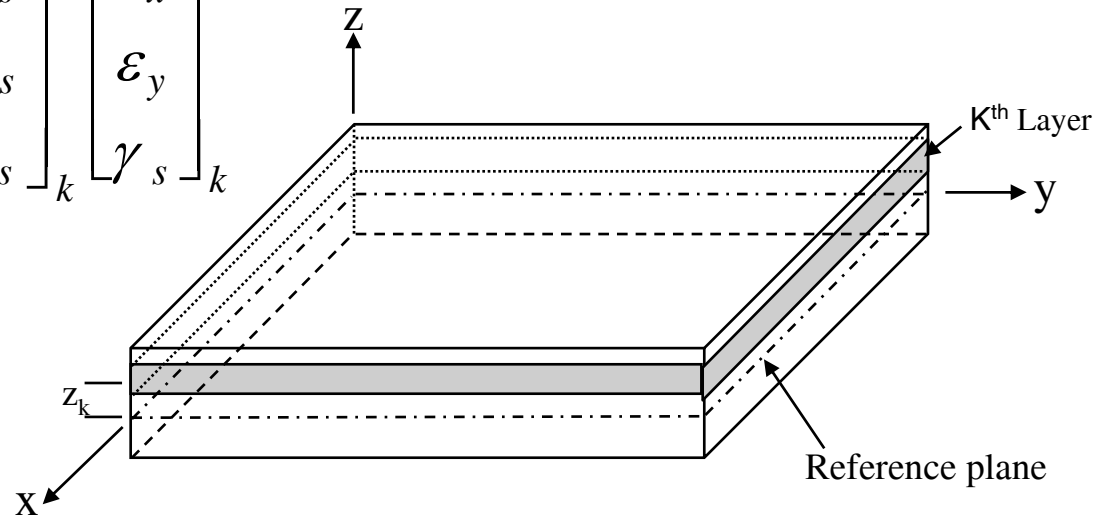


Layered Material



1.3 Stress-Strain Relations of an kth Layer in a Laminate

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix}_k = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_s \end{bmatrix}_k$$

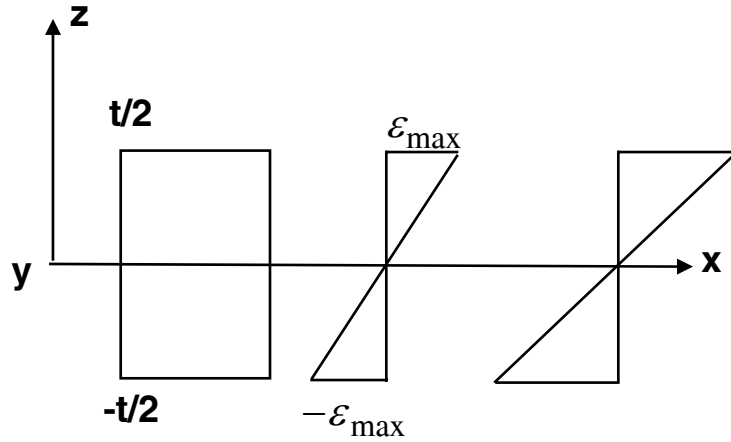


$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix}_k = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{bmatrix}_k + z \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}_k$$

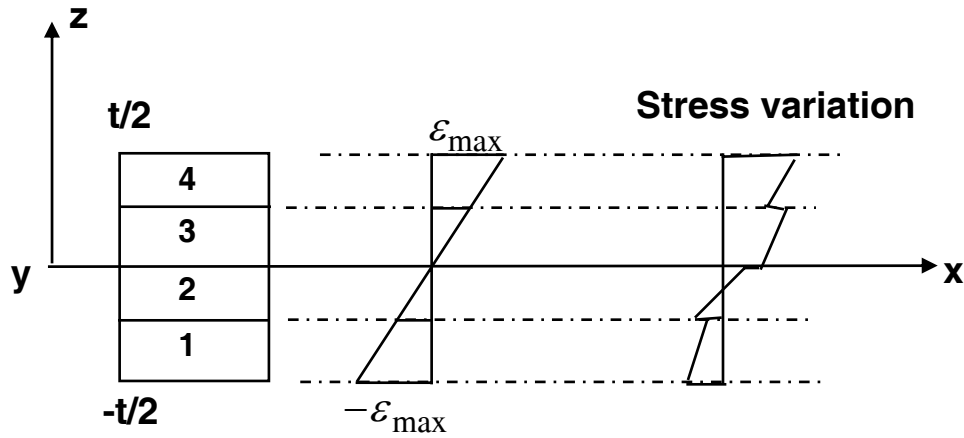
In short:

$$\{\sigma\}_{x,y}^k = [Q]_{x,y}^k \{\varepsilon^0\}_{x,y} + z [Q]_{x,y}^k \{\kappa\}_{x,y}$$

Isotropic/Single Layered Material



Layered Material

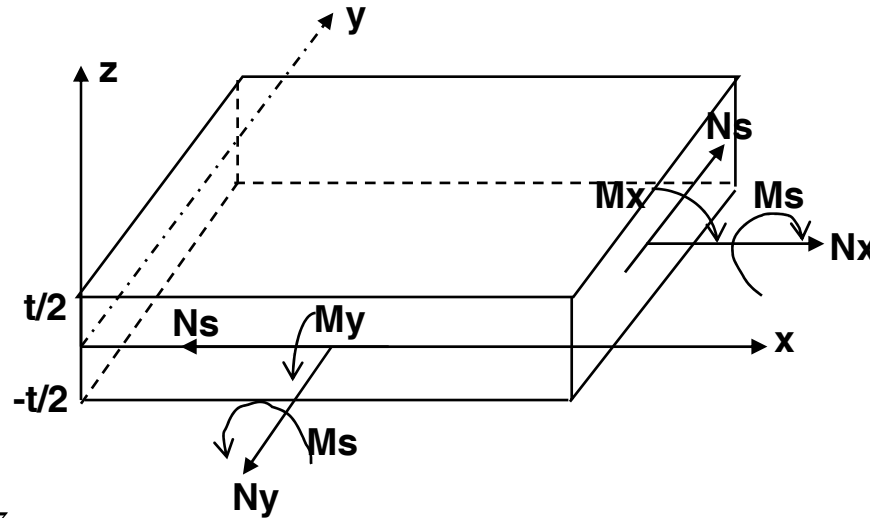


1.4 Force and Moment Resultants

$$N_x^k = \int_{-t/2}^{t/2} \sigma_x dz$$

$$N_y^k = \int_{-t/2}^{t/2} \sigma_y dz$$

$$N_{xy}^k = N_s^k = \int_{-t/2}^{t/2} \tau_s dz$$



$$M_x^k = \int_{-t/2}^{t/2} \sigma_x z dz$$

$$M_y^k = \int_{-t/2}^{t/2} \sigma_y z dz$$

$$M_{xy}^k = M_s^k = \int_{-t/2}^{t/2} \tau_s z dz$$

For layered materials:

$$\begin{bmatrix} N_x \\ N_y \\ N_s \end{bmatrix} = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix}_k dz$$

$$\begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{bmatrix}_k z dz$$

$$\begin{bmatrix} N_x \\ N_y \\ N_s \end{bmatrix} = \sum_{k=1}^n \left\{ \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{bmatrix}_k \int_{h_{k-1}}^{h_k} dz + \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z dz \right\}$$

$$\begin{aligned} [N]_{x,y} &= \left[\sum_{k=1}^n [Q]_{x,y}^k \int_{h_{k-1}}^{h_k} dz \right] [\varepsilon^0]_{x,y} + \left[\sum_{k=1}^n [Q]_{x,y}^k \int_{h_{k-1}}^{h_k} z dz \right] [\kappa]_{x,y} \\ &= \left[\sum_{k=1}^n [Q]_{x,y}^k (h_k - h_{k-1}) \right] [\varepsilon^0]_{x,y} + \left[\frac{1}{2} \sum_{k=1}^n [Q]_{x,y}^k (h_k^2 - h_{k-1}^2) \right] [\kappa]_{x,y} \\ &= [A]_{x,y} [\varepsilon^0]_{x,y} + [B]_{x,y} [\kappa]_{x,y} \end{aligned}$$

Where:

$$A_{i,j} = \sum_{k=1}^n [Q]_{i,j}^k (h_k - h_{k-1})$$

Extensional stiffness. It relates in-plane loads to in-plane strains.

$$B_{i,j} = \frac{1}{2} \sum_{k=1}^n [Q]_{i,j}^k (h_k^2 - h_{k-1}^2)$$

Coupling stiffness or in-plane/flexure coupling laminate moduli. It relates in-plane loads to curvatures and moments to in-plane strains.

$$\begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} = \sum_{k=1}^n \left\{ \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z dz + \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z^2 dz \right\}$$

$$\begin{aligned} [M]_{x,y} &= \left[\frac{1}{2} \sum_{k=1}^n [Q]_{x,y}^k (h_k^2 - h_{k-1}^2) \right] [\varepsilon^0]_{x,y} + \left[\frac{1}{3} \sum_{k=1}^n [Q]_{x,y}^k (h_k^3 - h_{k-1}^3) \right] [\kappa]_{x,y} \\ &= [B]_{x,y} [\varepsilon^0]_{x,y} + [D]_{x,y} [\kappa]_{x,y} \end{aligned}$$

$$A_{i,j} = \sum_{k=1}^n [Q]_{i,j}^k (h_k - h_{k-1})$$

$$B_{i,j} = \frac{1}{2} \sum_{k=1}^n [Q]_{i,j}^k (h_k^2 - h_{k-1}^2)$$

$$D_{i,j} = \frac{1}{3} \sum_{k=1}^n [Q]_{i,j}^k (h_k^3 - h_{k-1}^3)$$

Coupling stiffness or in-plane/flexure coupling laminate moduli. It relates in-plane loads to curvatures and moments to in-plane strains.

Bending or flexural stiffness. It relates moments to curvatures.

1.5.1 Load-Displacement Equation:

$$\begin{bmatrix} N_x \\ N_y \\ N_s \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} \\ A_{yx} & A_{yy} & A_{ys} \\ A_{sx} & A_{sy} & A_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{bmatrix} + \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{yx} & B_{yy} & B_{ys} \\ B_{sx} & B_{sy} & B_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{yx} & B_{yy} & B_{ys} \\ B_{sx} & B_{sy} & B_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{bmatrix} + \begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{yx} & D_{yy} & D_{ys} \\ D_{sx} & D_{sy} & D_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}$$

$$\begin{bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} & B_{xx} & B_{xy} & B_{xs} \\ A_{yx} & A_{yy} & A_{ys} & B_{yx} & B_{yy} & B_{ys} \\ A_{sx} & A_{sy} & A_{ss} & B_{sx} & B_{sy} & B_{ss} \\ \hline B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & D_{xs} \\ B_{yx} & B_{yy} & B_{ys} & D_{yx} & D_{yy} & D_{ys} \\ B_{sx} & B_{sy} & B_{ss} & D_{sx} & D_{sy} & D_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \\ \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix}$$

1.5.2 Displacement-Load Equation:

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \\ K_x \\ K_y \\ K_s \end{bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xs} & b_{xx} & b_{xy} & b_{xs} \\ a_{yx} & a_{yy} & a_{ys} & b_{yx} & b_{yy} & b_{ys} \\ a_{sx} & a_{sy} & a_{ss} & b_{sx} & b_{sy} & b_{ss} \\ \hline c_{xx} & c_{xy} & c_{xs} & d_{xx} & d_{xy} & d_{xs} \\ c_{yx} & c_{yy} & c_{ys} & d_{yx} & d_{yy} & d_{ys} \\ c_{sx} & c_{sy} & c_{ss} & d_{sx} & d_{sy} & d_{ss} \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} \varepsilon^0 \\ K \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix}$$

Where:

$$[a] = [A^{-1}] - \{ [B] [D^{*-1}] \} [C^*]$$

$$[b] = [B^*] [D^{*-1}]$$

$$[c] = -[D^{*-1}] [C^*]$$

$$[d] = [D^{*-1}]$$

$$[A^{-1}] = \text{inverse of matrix } [A]$$

$$[B^*] = -[A^{-1}] [B]$$

$$[C^*] = [B] [A^{-1}]$$

$$[D^*] = [D] - \{ [B] [A^{-1}] \} [B]$$

1.6 Special Class of Laminates

Laminate Staking Sequence: (angle&thickness/. /. /. /angle N&thickness N)

Constant ply thickness laminate: Ply thickness h (0.005±.0005')

$(0_2/45_5/90_2/45_5/0_2)$

Special Laminates:

Regular - Ply thickness is constant

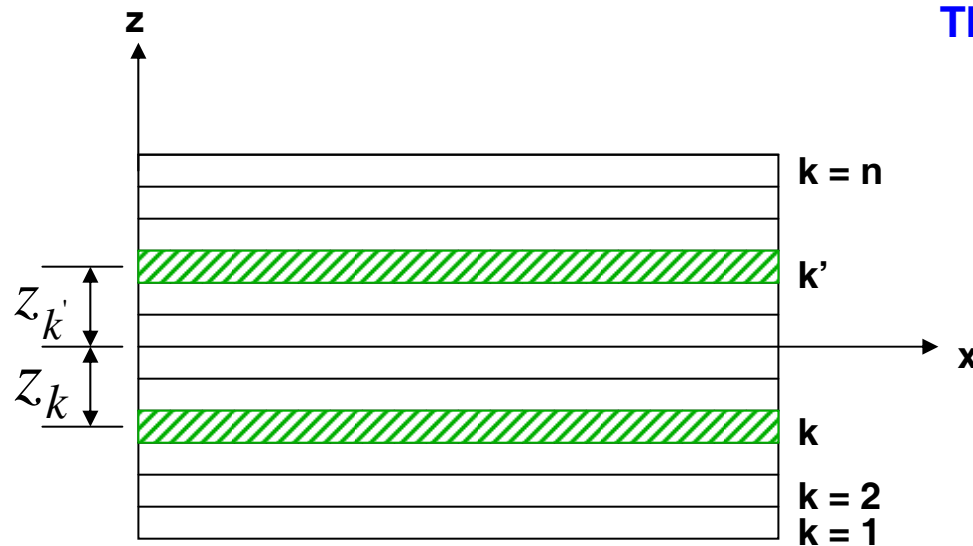
NOTE: Add more discussion on Symmetric and Balanced laminates

Symmetric Laminates

A laminate in which for each layer on one side of a reference plane there is an identical layer on the opposite of the reference plane at equal distance with same thickness, material properties, and orientation.

Symmetric definition requires symmetry of both geometry and material properties.

Example: $(0_2/45_5/90_2/45_5/0_2) = (0_2/45_5/90)_s$



Then

$$\begin{aligned}
 B_{ij} &= \frac{1}{2} \sum_{k=1}^n Q_{ij}^k (h_k^2 - h_{k-1}^2) \\
 &= \frac{1}{2} \sum_{k=1}^n Q_{ij}^k (h_k + h_{k-1})(h_k - h_{k-1}) \\
 &= \sum_{k=1}^n Q_{ij}^k z_k t_k
 \end{aligned}$$

$$t_k = t_{k'}$$

$$z_k = -z_{k'}$$

$$Q_{ij}^k = Q_{ij}^{k'} \quad (i, j = x, y, s)$$

Where

$$z_k = \frac{1}{2} (h_k + h_{k-1})$$

$$t_k = h_k - h_{k-1}$$

$$B_{ij} = 0, (i, j = x, y, s)$$

The load-deformation relations become:

$$\begin{bmatrix} N_x \\ N_y \\ N_s \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} \\ A_{yx} & A_{yy} & A_{ys} \\ A_{sx} & A_{sy} & A_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_s^o \end{bmatrix}$$

and

$$\begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} = \begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{yx} & D_{yy} & D_{ys} \\ D_{sx} & D_{sy} & D_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}$$

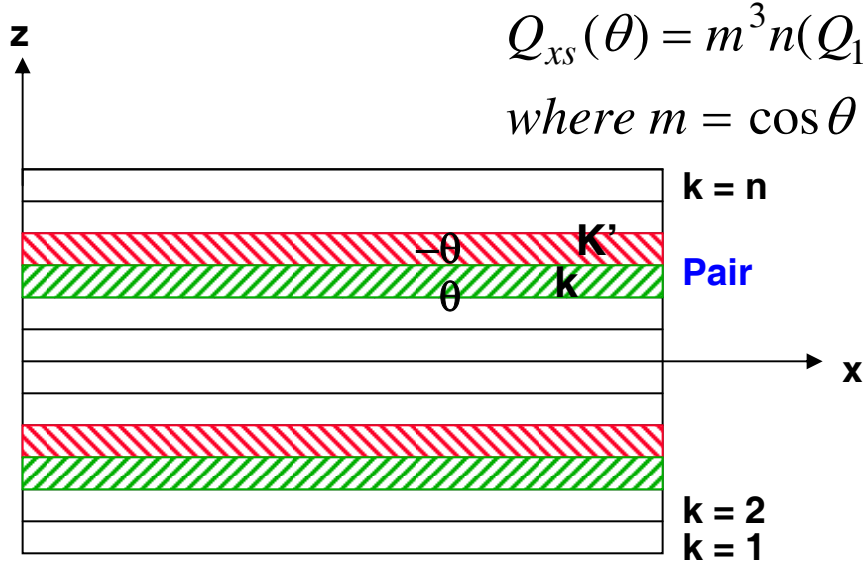
Special Cases:

- a. isotropic layers**
- b. specially orthotropic layers**
- c. Angle-Ply layers**

Balanced Laminate:

A laminate is balanced when it consists of **pairs of layers** with identical thickness and elastic properties but have $+\theta/-\theta$ orientation of their principal material properties with respect to the laminate reference axes.

(15₂/45₅/30/-30/-45₅/-15₂)



$$Q_{xs}(\theta) = m^3 n(Q_{11} - Q_{12} - 2Q_{66}) + mn^3(Q_{12} - Q_{22} + 2Q_{66})$$

where $m = \cos \theta$ and $n = \sin \theta$.

$$Q_{is}(\theta) = -Q_{is}(-\theta)$$

For each balanced pair of layers k and k'

$$t_k = t_{k'}$$

$$\theta_k = -\theta_{k'}$$

$$A_{is} = \sum_{k=1}^n Q_{is}^k (h_k - h_{k-1}) = \sum_{k=1}^n Q_{is}^k t_k$$

Where $l = x, y$

Therefore, for each pair $A_{is} = 0$ ($l=x,y$)

Types of Balanced Laminates:

Symmetric: $[\pm\theta_1/\pm\theta_2]_s$
Antisymmetric: $[\theta_1/\theta_2/-\theta_2/-\theta_1]$
Asymmetric: $[\theta_1/\theta_2/-\theta_1/-\theta_2]$

b. Antisymmetric Laminate

$$D_{is} = \frac{1}{3} \sum_{k=1}^n Q_{is}^k (h_k^3 - h_{k-1}^3) = 0$$

$$(h_k^3 - h_{k-1}^3) = (h_k^3 - h_{k-1}^3)$$

$$Q_{is}^k = -Q_{is}^k$$

$$\begin{bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & 0 & B_{xx} & B_{xy} & B_{xs} \\ A_{yx} & A_{yy} & 0 & B_{yx} & B_{yy} & B_{ys} \\ 0 & 0 & A_{ss} & B_{sx} & B_{sy} & B_{ss} \\ \hline B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & 0 \\ B_{yx} & B_{yy} & B_{ys} & D_{yx} & D_{yy} & 0 \\ B_{sx} & B_{sy} & B_{ss} & 0 & 0 & D_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_s^o \\ \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}$$

c. Antisymmetric Crossply Laminate (0/90)_n

$$z_k = -z_{k'}$$

$$t_k = t_{k'}$$

$$Q_{xx}^k = Q_{yy}^{k'}$$

$$Q_{yy}^k = Q_{xx}^{k'}$$

$$Q_{xy}^k = Q_{xy}^{k'}$$

$$Q_{xs}^k = Q_{ys}^k = Q_{xs}^{k'} = Q_{ys}^{k'} = 0$$

$$A_{xx} = A_{yy}$$

$$A_{xs} = A_{ys} = 0$$

$$B_{xx} = -B_{yy}$$

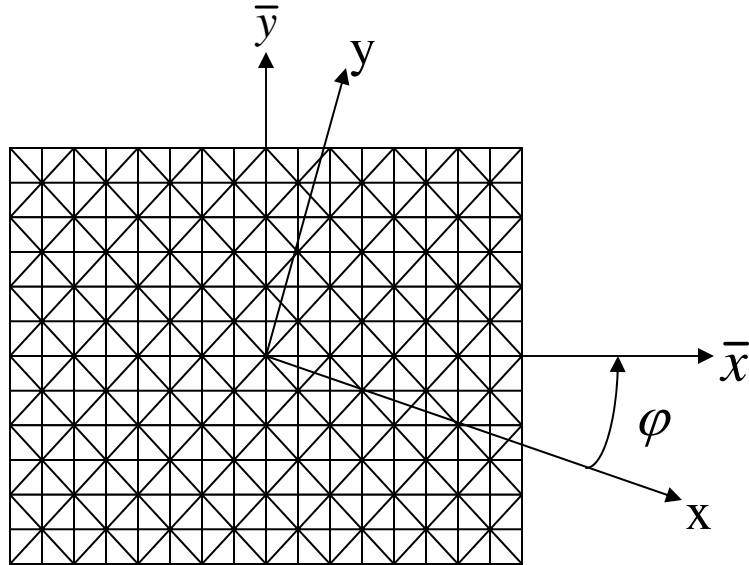
$$B_{xy} = B_{xs} = B_{ys} = B_{ss} = 0$$

$$D_{xx} = D_{yy}$$

$$D_{xs} = D_{ys} = 0$$

$$\begin{bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & 0 & B_{xx} & 0 & 0 \\ A_{yx} & A_{xx} & 0 & 0 & -B_{xx} & 0 \\ 0 & 0 & A_{ss} & 0 & 0 & 0 \\ \hline B_{xx} & 0 & 0 & D_{xx} & D_{xy} & 0 \\ 0 & -B_{xx} & 0 & D_{yx} & D_{xx} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_s^o \\ \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix}$$

Quasi-Isotropic Laminates



$$[A]_{\bar{x},\bar{y}} = [A]_{x,y} = \text{constant}$$

$$[a]_{\bar{x},\bar{y}} = [a]_{x,y} = \text{constant}$$

$$\bar{E}_{\bar{x}} = \bar{E}_x = \text{constant}$$

$$\bar{G}_{\bar{x}\bar{y}} = \bar{G}_{xy} = \text{constant}$$

$$\bar{\nu}_{\bar{x}\bar{y}} = \bar{\nu}_{xy} = \text{constant}$$

$$\bar{\eta}_{\bar{x}\bar{s}} = \bar{\eta}_{\bar{y}\bar{s}} = \bar{\eta}_{xs} = \bar{\eta}_{ys} = 0$$

$$\left[0 / \frac{\pi}{n} / \frac{2\pi}{n} / \dots / \frac{n-1}{n} \pi \right]_s \quad \text{or} \quad \left[\frac{\pi}{n} / \frac{2\pi}{n} / \dots / \pi \right]_s$$

Lay-up is quasi-isotropic for any integer n greater than 2.

Other quasi-isotropic layups: $[0/\pm 45/90]_s$
 $[0/60/-60]_s$

Summary

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1.4 Stress Resultants

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Analysis Of Laminated Composite Structures

