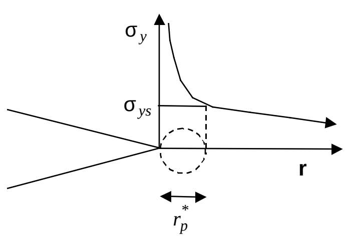
Chapter 4: Crack Tip Plastic Zone

4.1 Irwin's Plastic Zone correction

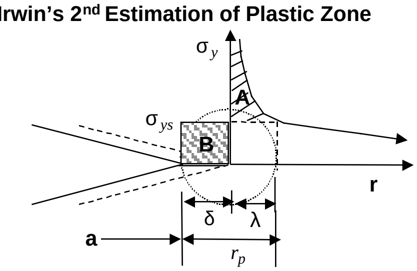
First Estimation of Plastic Zone:



$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \qquad \sigma_{ys} = \frac{K_I}{\sqrt{2\pi r_p^*}}$$

$$r_p^* = \frac{K_I^2}{2\pi\sigma_{ys}^2} = \frac{a}{2}\left(\frac{\sigma}{\sigma_{ys}}\right)^2$$

Irwin's 2nd Estimation of Plastic Zone



$$\sigma_{ys} = \frac{K_I}{\sqrt{2\pi\lambda}} = \sigma\sqrt{\frac{a+\delta}{2\lambda}}$$

or

$$\lambda = \frac{\sigma^2(a+\delta)}{2\sigma_{ys}^2} \approx r_p^*$$

$$\delta\sigma_{ys} = \left[\int_0^{\lambda} \sigma\sqrt{\frac{a+\delta}{2r}}\,dr\right] - \lambda\sigma_{ys}$$

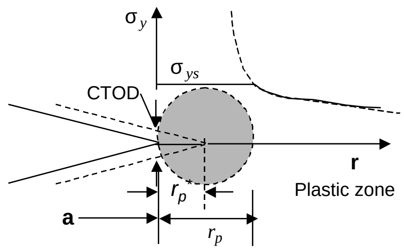
Neglecting δ as compared to *a* and substituting for λ

$$(\delta + r_p^*)\sigma_{ys} = \sigma\sqrt{2ar_p^*} \quad \textbf{or} \quad (\delta + r_p^*)^2 = \frac{2\sigma^2}{\sigma_{ys}^2}ar_p^* = 4r_p^{*2}$$

Hence

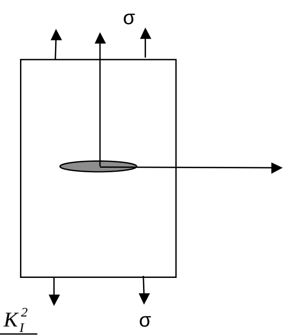
$$\delta = r_p^* \quad \textit{and} \quad r_p = \lambda + \delta = 2r_p^*$$

Irwin's Plastic zone correction



Effective SIF

$$K_I = \beta\sigma\sqrt{\pi(a + r_p^*)} = \beta\sigma\sqrt{\pi\left(a + \frac{K_I^2}{2\pi\sigma_{ys}^2}\right)}$$



Crack Tip Opening Displacement

Recall expression for COD:

$$COD = 2v = \frac{4\sigma}{E}\sqrt{a^2 - x^2}$$

$$COD = \frac{4\sigma}{E}\sqrt{\left(a + r_p^*\right)^2 - x^2}$$

CTOD is found for x=a,

$$CTOD = \frac{4\sigma}{E}\sqrt{\left(a + r_p^*\right)^2 - a^2} \approx \frac{4\sigma}{E}\sqrt{2ar_p^*} = \frac{4}{\pi}\frac{K_I^2}{E\sigma_{ys}}$$

The Dugdale and Barenblatt Approach (Strip Yield model)

Assumption:

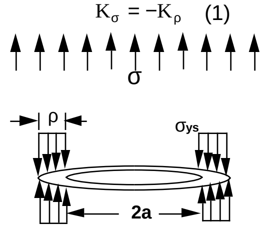
1. Effective crack length is larger than the physical crack.

2. The crack edges, ρ, in front of the physical crack carry the σ_{ys}, tending to close the crack.

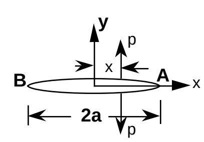
3. The size ρ is chosen such that the SIF due to the combined loading at the effective crack tip is zero.

$K_{\text{due to remote stress } \sigma} + K_{\text{due to wedge force } \sigma ys} = 0$ **at x=a+ρ.**

$$K_\sigma = -K_\rho \quad (1)$$



Dugdale Crack Tip Model



Wedge force solution

Wedge Solution:

$$K_A = \frac{p}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}} \qquad K_B = \frac{p}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}}$$

K @ A & B due to distributed loadfrom *s* to the crack tip *a* is

$$K = \frac{p}{\sqrt{\pi a}}\int_s^a \left\{\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}}\right\}dx$$

Integrating the above equation, and then substituting

s* = *a

***a* = *a* + ρ and**

***p* = σ_{ys} , we get**

$$K_\rho = 2\sigma_{ys}\sqrt{\frac{a+\rho}{\pi}}\arccos\frac{a}{a+\rho} \qquad (2)$$

SIF Due to Remote Stress

$$K_\sigma = \sigma\sqrt{\pi(a+\rho)} \qquad (3)$$

Substituting Eqs. 2 and 3 in Eq. 1, we can get an equation for ρ,

$$\rho = \frac{\pi^2 a}{8}\left(\frac{\sigma}{\sigma_{ys}}\right)^2 = \frac{\pi}{8}\left(\frac{K}{\sigma_{ys}}\right)^2$$

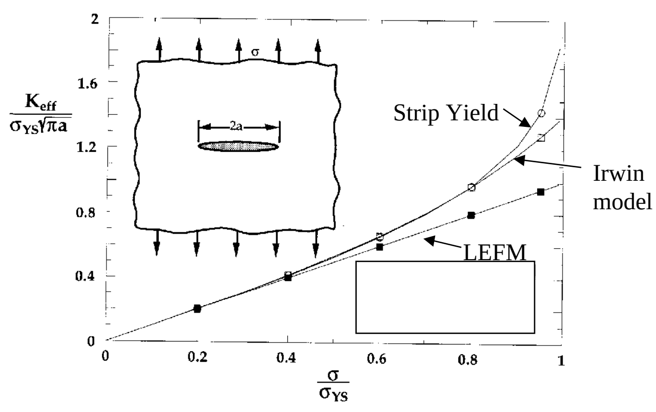
While Irwin's plastic zone size:

$$\rho = 2r_p^* = a\left(\frac{\sigma}{\sigma_{ys}}\right)^2 = \frac{1}{\pi}\left(\frac{K}{\sigma_{ys}}\right)^2$$

The two solution are almost identical for small scale yielding. The difference is large for large values of (σ/σ_{ys}).

Note: Both Irwin and Dugdale solutions have limited validity for large scale yielding, because of limitations of linear elastic K solution.

Comparison of Irwin and Dugdale Plastic Zone Corrections Through-Crack, Plane-Strain Condition



4.4 Plastic ZoneShapes

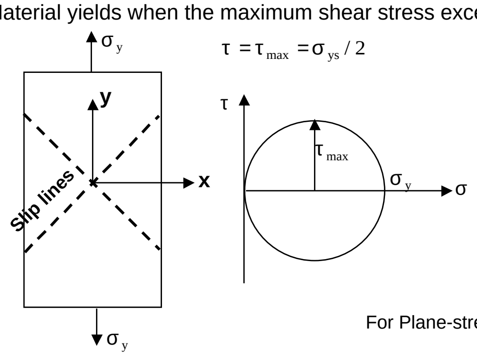
4.4.1 Yield Criterion

Let us examine the plastic zone based on the elastic stress field alone.
That means, redistribution of stresses due to the yielding is neglected.
Two yield criteria are used in metallic materials:

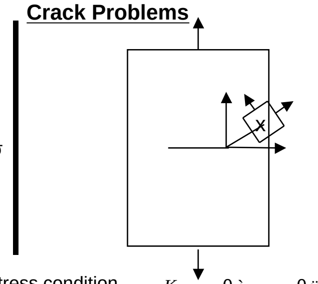
- Tresca
- vonMises

1. Tresca Yield Criterion

Material yields when the maximum shear stress exceeds the yield shear strength.



Crack Problems



For Plane-stress condition

$$\sigma_1 = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left\{ 1 + \sin\frac{\theta}{2} \right\}$$

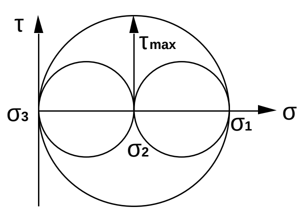
$$\sigma_2 = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left\{ 1 - \sin\frac{\theta}{2} \right\}$$

For Plane-strain condition

$$\sigma_3 = \nu(\sigma_1 + \sigma_2) = 2\nu \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

4.4.1. Plastic zone based onTresca yield criterion

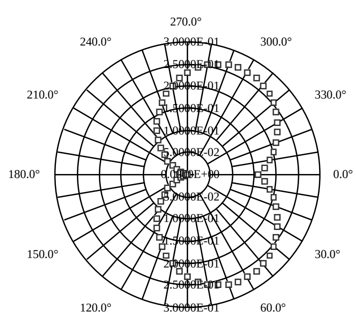
Plane Stress, $\sigma_3 = 0$



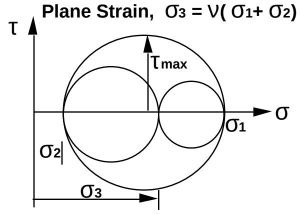
$$\tau_{max} = \frac{(\sigma_1 - \sigma_3)}{2} = \frac{\sigma_1}{2}$$

$$2\tau_{max} = \sigma_1 = \sigma_{ys} = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left\{ 1 + \sin\frac{\theta}{2} \right\}$$

$$r_p(\theta) = \frac{K^2}{2\pi\sigma_{ys}^2} \left[\cos\frac{\theta}{2} \left\{ 1 + \sin\frac{\theta}{2} \right\} \right]^2$$



Plane Strain, $\sigma_3 = \nu(\sigma_1 + \sigma_2)$

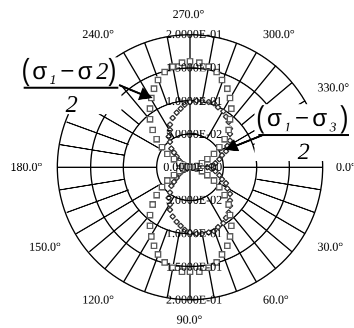


$$\tau_{max} = \frac{(\sigma_1 - \sigma_2)}{2} \quad or \quad \frac{(\sigma_1 - \sigma_3)}{2}$$

$$2\tau_{max} = \sigma_1 - \sigma_2 = \sigma_{ys} = \frac{2K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{K}{\sqrt{2\pi r}}\sin\theta \qquad 2\tau_{max} = \sigma_1 - \sigma_3 = \sigma_{ys} = \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left\{1 - 2\nu + \sin\frac{\theta}{2}\right\}$$

$$r_p = \frac{K^2}{2\pi\sigma_{ys}^2}\sin^2\theta \qquad r_p(\theta) = \frac{K^2}{2\pi\sigma_{ys}^2}\left[\cos\frac{\theta}{2}\left\{1 - 2\nu + \sin\frac{\theta}{2}\right\}\right]^2$$

Note: All three expressions arefunctions of r_p*, Irwins's first estimation of plastic zone.



2. von Mises Yield Criterion:

Material yields when the octahedral shear stress exceeds the maximum shear yield strength. von Mises criterion in terms of the principal stresses is

$$\sigma_{eff} = \frac{1}{\sqrt{2}}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}^{1/2} \geq \sigma_{ys}$$

4.4.2 Plastic Zone Based on von Mises Criterion

Plane-stress

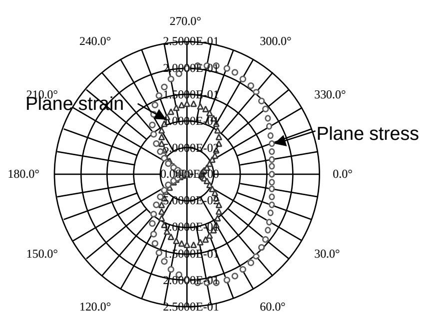
$$\sigma_{ys}^2 = \frac{K^2}{4\pi r}\left\{1 + \frac{3}{2}\sin^2\theta + \cos\theta\right\}$$

$$r_p(\theta) = \frac{K^2}{4\pi\sigma_{ys}^2}\left\{1 + \frac{3}{2}\sin^2\theta + \cos\theta\right\}$$

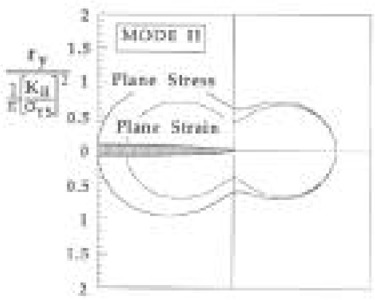
Plane-Strain

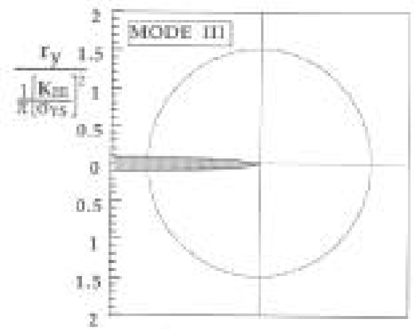
$$\sigma_{ys}^2 = \frac{K^2}{4\pi r}\left\{\frac{3}{2}\sin^2\theta + (1 - 2\nu)^2(1 + \cos\theta)\right\}$$

$$r_p(\theta) = \frac{K^2}{4\pi\sigma_{ys}^2}\left\{\frac{3}{2}\sin^2\theta + (1 - 2\nu)^2(1 + \cos\theta)\right\}$$



Von Mises Elastic Yield Zones for Mode II & III Loads





4.5 Realistic Plastic Zones

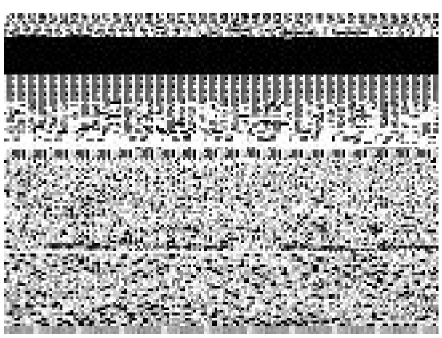
In deriving the plastic zones in the previous sections, the redistribution of stresses due to yielding was neglected. This introduces error in estimating the size and shape of the plastic zone. A number of relaxation and elastic-plastic studies have been made to define the realistic palstic zones. Among these studies, the oldest and accurate solutions were by

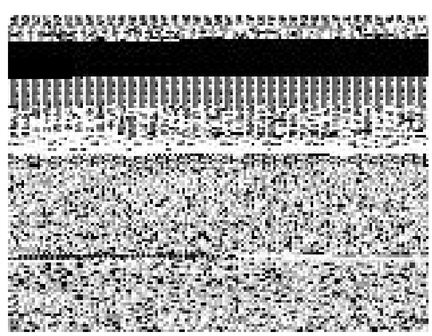
Tuba (1966)	**Elastic-plastic materials**
Rice & Rosengren(1968) --	**Strain hardening materials**
Hahn & Rsenfield (1965-68)	**Experimental data**
Dodds (1990's)	

$$\frac{\varepsilon}{\varepsilon_o} = \frac{\sigma}{\sigma_o} + \alpha\left(\frac{\sigma}{\sigma_o}\right)^n, \quad \varepsilon_o,\ \sigma_o,\ \alpha, \textit{ and n- materials constants}$$

σ_e − *von Mises effective stress*

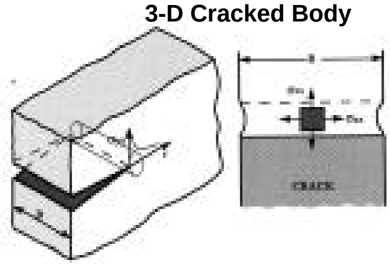
σ_{ys} *0.2% offset yield stress*



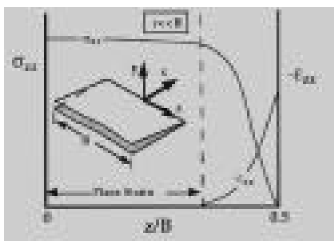


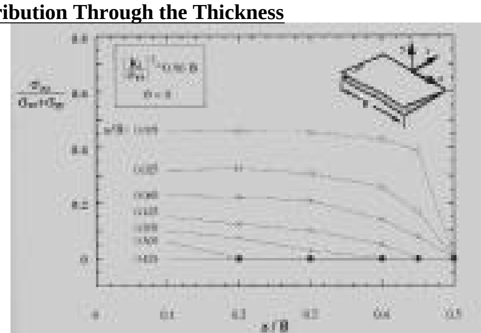
4.6 Plane Stress & Plane Strain Regions in Cracked Bodies

3-D Cracked Body



Transverse Stress Distribution Through the Thickness

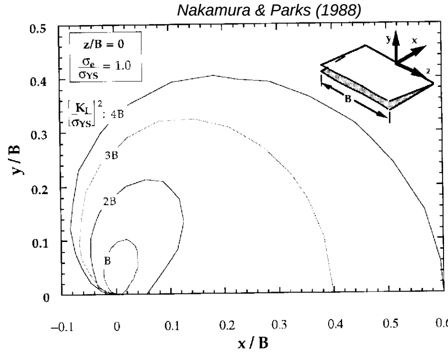




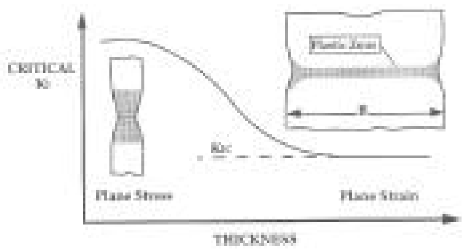
Narasimhan & Rasakis (1988)

Effect of K_I/B on Plastic Zone Size and Shape

Nakamura & Parks (1988)



Variation of Fracture Toughness With Specimen Thickness



4.7 Plastic Constraint Factor

$$pcf = \frac{\sigma_{max}}{\sigma_{ys}}$$

The quantity p.c.f * σ_{ys}, can be considered as an effective yield stress.

<u>Estimation of p.c.f:</u>

Let us assume

$$\sigma_2 = n\sigma_1 \quad \sigma_3 = m\sigma_1$$

Substituting in the von Mises criterion we get

$$pcf = \frac{\sigma_1}{\sigma_{ys}} = \frac{1}{\sqrt{\left[1 - n - m + n^2 + m^2 - mn\right]}}$$

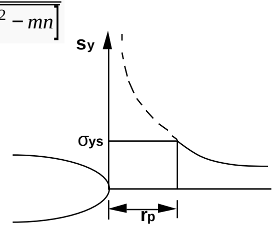
Plane stress: *pcf* = 1

Plane strain: n=1 & m=2ν

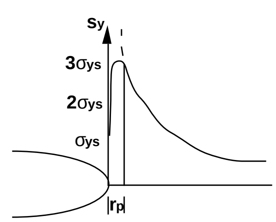
For ν=1/3, **pcf** = 3

Irwin's plane strain plastic zone

$$r_p^* = \frac{K^2}{2\pi(3\sigma_{ys})^2} = \frac{K^2}{18\pi\sigma_{ys}^2}$$



Plane-stress condition

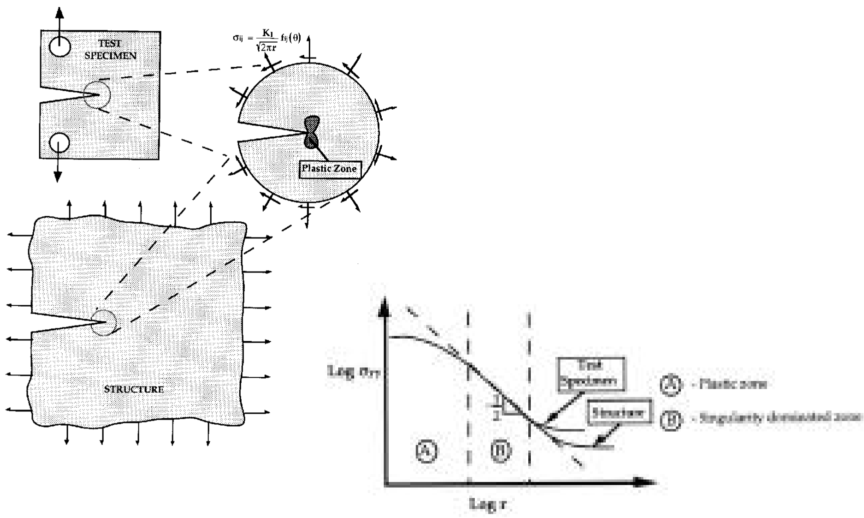


Plane-strain condition

Because the constraint varies through the thickness and ahead of the crack tip, Irwin used an average constraint as , $\sqrt{2\sqrt{2}} = 1.68$ which modifies the r*p equation into

$$r_p^* = \frac{K^2}{2\pi(1.68\sigma_{ys})^2} = \frac{K^2}{6\pi\sigma_{ys}^2}$$

K Controlled Fracture in Test Specimen and Structure



4.7 Summary

1. Irwin's plastic zone correction , $r_p^* = \dfrac{K_I^2}{2\pi\sigma_{ys}^2}$, Plane-stress condition

Plane-strain $r_p^* = \dfrac{K_I^2}{6\pi\sigma_{ys}^2}$

2. Plastic zone size- Irwin's $r_p = 2r_p^* = \dfrac{K_I^2}{\pi\sigma_{ys}^2}$

Dugdale-Barenblatt: $r_p = \dfrac{\pi}{8}\dfrac{K_I^2}{\sigma_{ys}^2}$

3. Effective SIF, $K_I = \sigma\sqrt{\pi\left(a + r_p^*\right)}$

4. CTOD = COD @ crack-tip $= \dfrac{4\sigma}{E}\sqrt{2ar_p^*} = \dfrac{4K_I^2}{\pi E\sigma_{ys}^2}$

5. Plastic zone size and shape depends on
   a. State of stress (plane stress or stain - panel thickness)
   b. Yield criteria

6. Plastic constraint factor, a measure of elevation of yield stress,

pcf = $\dfrac{\sigma_{max}}{\sigma_{ys}}$; pcf = 1 for Plane-stress

pcf = 3 for plane-strain